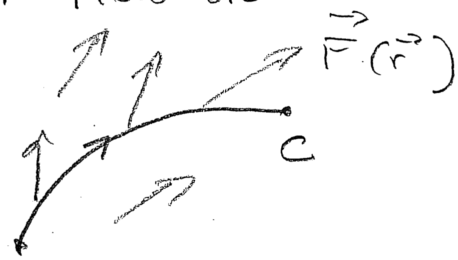
S10 Apr 20, 27

18.1 Line Integrals

Let a vector field $\vec{F}(\vec{r})$ and an oriented curve C on the xy-plane we given. "Oriented" means that we have a direction of motion on C. How do we define the line integral:

$$\int_C \vec{F}(\vec{r})\, d\vec{r} \quad ?$$

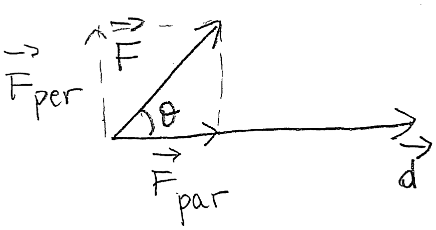


We define it in such a way that if $\vec{F}(\vec{r})$ is a force-field, then the integral represents the work done be the force as the body moves along C. What definition will capture that?

Recall the basic formula:

$$\text{Work} = \text{Force} \times \text{Distance}$$

which is valid provided an object moves along a directected segment in the exact direction of the force. What if the displacement and the force are not parallel:



$$\text{Work} = \|\vec{F}_{par}\| \cdot \|\vec{d}\|$$

$$\|\vec{F}_{per}\| = \|\vec{F}\| \cdot \cos\theta$$

The dot product.

So: $\underline{\text{Work}} = \|\vec{F}\| \cdot \|\vec{d}\| \cdot \cos\theta = \underline{\vec{F} \cdot \vec{d}}$

Work is the dot product of the force and displacement.

Now we will define

$$\int_C \vec{F}(\vec{r})\, d\vec{r}$$

in such a way that the integral represent work.

We divide C into very small pieces by choosing points $\vec{r}_0, \vec{r}_1, ---, \vec{r}_n$ along C. If C is smooth each piece is almost a segment so the displacement from $\vec{r}_{i-1}$ to $\vec{r}_i$ along C practically coincides with the displacement along the vector $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{i-1}$. $\vec{F}(\vec{r})$ changes along a small piece very little as well (if $\vec{F}(\vec{r})$ is continuous). Thus:

$$\text{Work over the } i\text{-th piece} \cong \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$$

$$\text{Total work} \cong \sum_i \vec{F}(\vec{r}_i)\, \Delta\vec{r}_i$$

We get better and better approximation as $\|\Delta\vec{r}_i\| \to 0$:

$$\text{Total work} = \lim_{\|\Delta\vec{r}_i\| \to 0} \sum_i \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i \triangleq \int_C \vec{F}(\vec{r})\, d\vec{r}$$

Under proper regularity assumptions about $\vec{F}$ and C (C smooth, $\vec{F}$ continuous) the limit exists and does not depend on the choice of $\vec{r}_i$. This limit is the line integral:

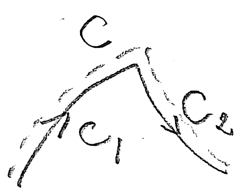
$$\int_C \vec{F}(\vec{r})d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

Physically the integral represents work. Mathematically it represents "how much" the vector field $\vec{F}$ "goes in the direction" of C, "along C":

$\vec{F}\cdot\Delta\vec{r}$

$\vec{F} \perp C \to \int_C \vec{F} = 0$

A few additional definitions and properties of the line integral:



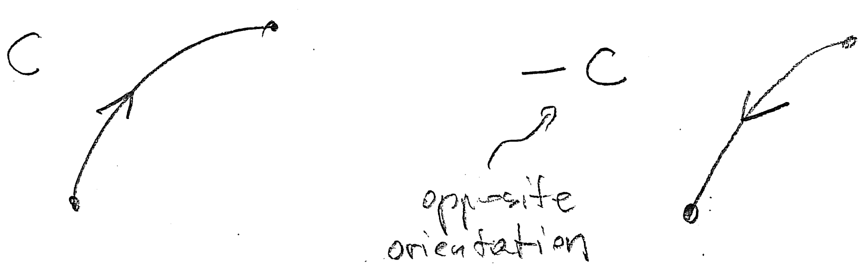
$C = C_1 + C_2$

↗ Notation

The definition of $\int_C \vec{F}d\vec{r}$

↙ can be extended to piecewise smooth path.

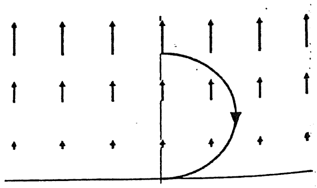
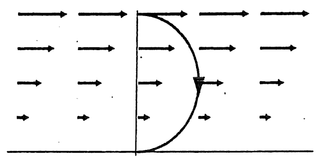
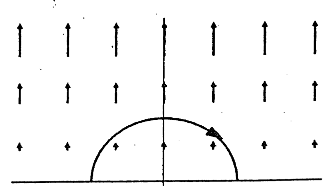
$$\int_C \vec{F}(\vec{r})d\vec{r} = \int_{C_1} \vec{F}(\vec{r})d\vec{r} + \int_{C_2} F(\vec{r})d\vec{r}$$



For a scalar constant λ, vector fields $\vec{F}$ and $\vec{G}$, and oriented curves C, C_1, and C_2

1. $\int_C \lambda\vec{F}\cdot d\vec{r} = \lambda\int_C \vec{F}\cdot d\vec{r}$.
2. $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$.
3. $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$.
4. $\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$.

Ex: Is $\int_C \vec{F}(\vec{r})d\vec{r}$ positive, negative or 0? C and $\vec{F}$ are depicted below.

(a)  (b)  (c) 

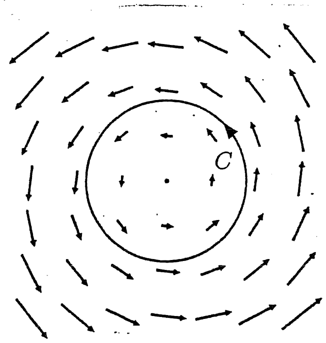
Negative. Positive. 0.

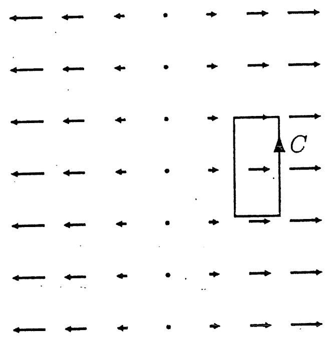
Def: If C is a closed oriented curve, then

$$\int_C \vec{F}(\vec{r})d\vec{r} = \oint_C \vec{F}(\vec{r})d\vec{r}$$

is called the circulation of $\vec{F}$ around C.

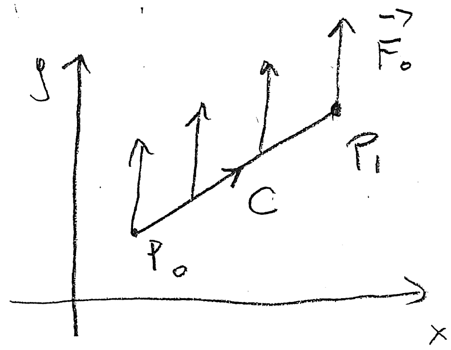
Ex: Find the sign of the circulation of the vector fields around the indicated paths.

(a) 

(b) 

Positive. Zero.

Remark: Let $\vec{F}(\vec{r}) = \vec{F}_0$ be constant. Let C be an oriented straight line segment from P_0 to P_1.



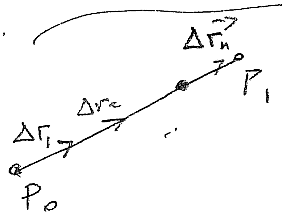
Then:

$$\int_C \vec{F}(\vec{r})\,d\vec{r} = \vec{F}_0 \cdot \overrightarrow{P_0P_1}$$

Follows easily from the definition of the line integral:

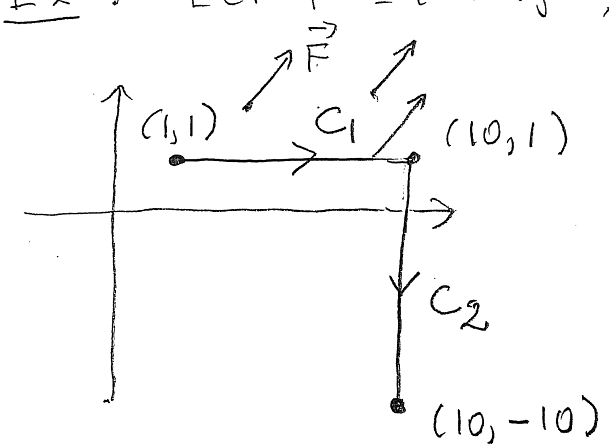
$$\int_C \vec{F}\,d\vec{r} = \lim_{\|\Delta\vec{r}_i\| \to 0} \sum_i \vec{F}(\vec{r}_i') \cdot \Delta\vec{r}_i = \lim_{\|\Delta\vec{r}_i\| \to 0} \sum_i \vec{F}_0 \cdot \Delta\vec{r}_i =$$

$$= \lim_{\|\Delta\vec{r}_i\| \to 0} \vec{F}_0 \cdot \left(\sum_i \Delta\vec{r}_i\right) = \lim_{\|\Delta\vec{r}_i\| \to 0} \vec{F}_0 \cdot \overrightarrow{P_0P_1} = \vec{F}_0 \cdot \overrightarrow{P_0P_1}$$



$$\sum_i \Delta\vec{r}_i = \overrightarrow{P_0P_1}$$

Ex: Let $\vec{F} = \vec{i} + 2\vec{j}$, C be the following path:



$$C = C_1 + C_2$$

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r} =$$

$$= (\vec{i} + 2\vec{j}) \cdot (9\vec{i}) +$$

$$+ (\vec{i} + 2\vec{j}) \cdot (-11\vec{j}) =$$

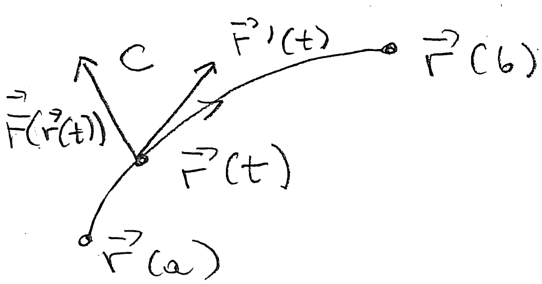
$$= 9 - 22 = \underline{-13}$$

In some simple cases we can find the line integral from the Remark or from the definition. What do we do in general? Parametrize C.

If $\vec{r}(t)$, for $a \leq t \leq b$, is a smooth parameterization of an oriented curve C and $\vec{F}$ is a vector field which is continuous on C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}\,'(t)$, of the parameterization of C, then integrate along the curve.



$$\int_C \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

dot product.

In terms of coordinates:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad t \text{ in } [a, b],$$

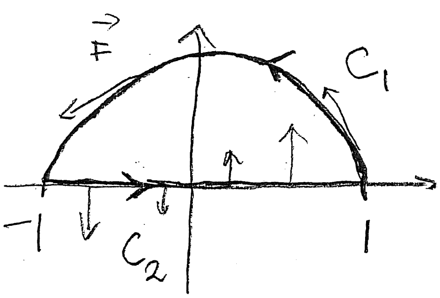
$$\vec{F}(\vec{r}(t)) = F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}$$

$$\int_C \vec{F}\, d\vec{r} = \int_a^b \left(F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}\right) \cdot \left(x'(t)\vec{i} + y'(t)\vec{j}\right) dt$$

Ex: Find $\int_C \vec{F}(\vec{r})\,d\vec{r}$ where:

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j},$$

C is given by $C = C_1 + C_2$:



C_1 - the upper hemicircle of the unit circle oriented ccw.

C_2 - the segment $(-1,0), (1,0)$.

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r}$$

Note $\vec{F} \perp C_2$ at each point along C_2. Thus $\int_{C_2} \vec{F} = 0$.

We can check by parametrization:

C_2: $\quad x(t) = t, \quad y(t) = 0, \quad t$ in $[-1, 1]$.

Thus:

$$\int_{C_2} \vec{F}(\vec{r})\,d\vec{r} = \int_{-1}^{1} F(x(t), y(t))] \cdot (x'(t)\vec{i} + y'(t)\vec{j})\,dt =$$

$$= \int_{-1}^{1} (t\vec{j}) \cdot (\vec{i})\,dt = \int_{-1}^{1} 0\,dt = 0.$$

Now C_1.

C_1: $\quad x(t) = \cos(t), \quad y(t) = \sin(t)$, t in $[0, \pi]$

or, in vector form:

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

$$\vec{r}\,'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}.$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = -\sin(t)\vec{i} + \cos(t)\vec{j},$$

Thus:

$$\int_{C_1} \vec{F}(\vec{r})\,d\vec{r} = \int_0^{\pi} (-\sin(t)\vec{i} + \cos(t)\vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\,dt =$$

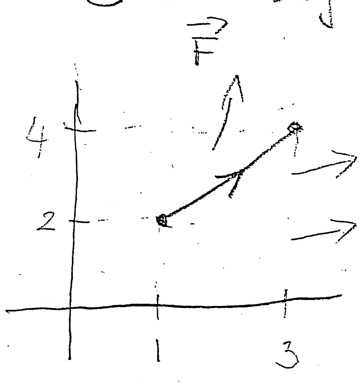
$$= \int_0^{\pi} \underbrace{(\sin^2(t) + \cos^2(t))}_{=1}\,dt = \pi.$$

Thus:

$$\int_C \vec{F}\,d\vec{r} = \pi$$

<u>Ex</u>. Find $\int_C \vec{F}$ where $\vec{F}(\vec{r}) = x^2\vec{i} + y^2\vec{j}$,

C the segment from $(1, 2)$ to $(3, 4)$.



Parametrize C:

$$\vec{w} = \overrightarrow{(1,2)(3,4)} = 2\vec{i} + 2\vec{j}$$

So we could take:

$x = 1 + 2t$, $\quad y = 2 + 2t$, $\quad t$ in $[0, 1]$.

We can take instead a simpler parametrization:

$$C: \quad x(t) = 1 + t, \quad y(t) = 2 + t, \quad t \text{ in } [0,2].$$

We have:

$$\vec{r}(t) = (1+t)\vec{i} + (2+t)\vec{j},$$

$$\vec{r}\,'(t) = \vec{i} + \vec{j}$$

$$\vec{F}(\vec{r}(t)) = (1+t)^2\vec{i} + (2+t)^2\vec{j}.$$

So:

$$\int_C \vec{F}\,d\vec{r} = \int_0^2 \left(\left((1+t)^2\vec{i} + (2+t)^2\vec{j}\right)\cdot(\vec{i}+\vec{j})\right)dt =$$

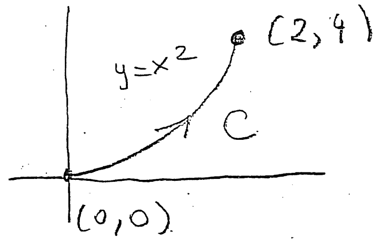
$$= \int_0^2 \left((1+t)^2 + (2+t)^2\right)dt = \int_0^2 (1+2t+t^2+4+4t+t^2)\,dt =$$

$$\int_0^2 (5+6t+2t^2)\,dt = \left(5t+3t^2+\tfrac{2}{3}t^3\right)\Big|_0^2 = \frac{82}{3}.$$

<u>Ex</u>: Find $\int_C F(\vec{r})\,d\vec{r}$ where

$$\vec{F}(x,y) = -y\sin x\,\vec{i} + \cos x\,\vec{j},$$

C is the piece of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.



C: $x(t)=t$, $y(t)=t^2$, t in $[0,2]$.

$$\vec{r}(t) = t\vec{i} + t^2\vec{j}, \quad \vec{r}\,'(t) = \vec{i} + 2t\vec{j}$$

$$\vec{F}(\vec{r}(t)) = -t^2\sin(t)\,\vec{i} + \cos(t)\,\vec{j}$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^2 \left(-t^2\sin(t) + 2t\cos(t)\right)dt =$$

$-\sin(t) = (\cos(t))'$ $\quad$ $2t = (t^2)'$

$$= t^2\cos(t)\Big|_0^2 = \underline{4\cos(2)}.$$

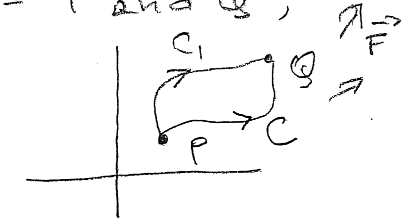
You can use your graphing calculator.

1) No quiz R
2) Final

Apr 21, 23

18.3 Path-independent Vector Fields

Def: A vector field $\vec{F}$ is said to be path-independent (or conservative), if for any two points P and Q, the line integral



$$\int_C \vec{F} d\vec{r}$$

is the same for any path C from P to Q.

If $\vec{F}$ is conservative, we can simply write:

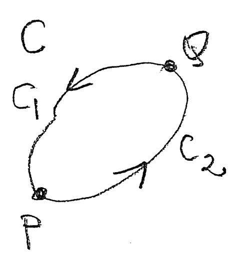
$$\int_P^Q \vec{F} d\vec{r}.$$

It seems like a strange property. Are there any p-i fields? We shall soon see. The first important observation:

Th: $\vec{F}$ is conservative if and only if for any closed path C:

$$\oint_C \vec{F} d\vec{r} = 0.$$

Indeed. "⇒" Let $\vec{F}$ be conservative and C be closed, choose and two points P and Q on C, take C_1 and C_2 as on the picture. Then:

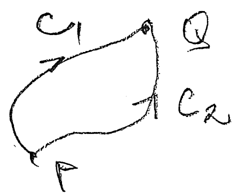


$$\int_P^Q \vec{F} d\vec{r} = \int_{C_2} \vec{F} d\vec{r} = \int_{-C_1} \vec{F} d\vec{r} = -\int_{C_1} \vec{F} d\vec{r},$$

So

$$\oint_C \vec{F} d\vec{r} = \int_{C_2} \vec{F} d\vec{r} - \left(-\int_{C_1} \vec{F} d\vec{r}\right) = 0.$$

Similarly "$\Leftarrow$". Suppose $\oint_C \vec{F}\,d\vec{r} = 0$ for any closed path C. To show that $\vec{F}$ is conservative take any P and Q and any two paths C_1, C_2 from P to Q:



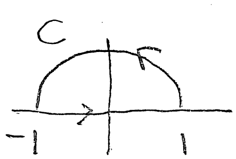
Then:

$$\int_{C_1} \vec{F} + \int_{-C_2} \vec{F} = 0 = \int_{C_1} \vec{F} - \int_{C_2} \vec{F} = 0$$

Thus:
$$\int_{C_1} \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r}.$$

So
$$\vec{F} \text{ conservative} \iff \oint_C \vec{F}\,d\vec{r} = 0 \text{ for any closed path.}$$

<u>Ex</u>: Recall: $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$



$$\oint_C \vec{F}\,d\vec{r} = \pi.$$

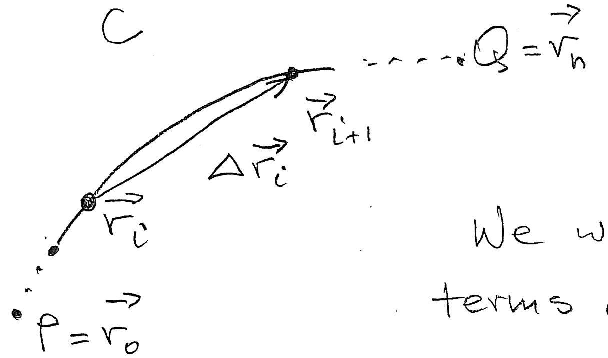
Thus $\vec{F}$ is not conservative.

So what vector fields are conservative? Gradient fields and only gradient fields. If

$$\vec{F}(x,y) = \nabla f(x,y)$$

for some $f(x,y)$, then $\vec{F}$ is conservative and vice-versa. Why? Let me give some justification.

Let $\vec{F}(x,y) = \nabla f(x,y)$ for some $f(x,y)$. Let C be an oriented smooth path from P to Q.



$$\int_C \vec{F}\, d\vec{r} = \lim_{\|\Delta\vec{r}\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

We will express $\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$ in terms of f. Observe:

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i)\cdot\|\Delta\vec{r}_i\|$$

The directional derivative at $\vec{r}_i$ in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$, i.e, the rate of change

magnitude of the displacement in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$

But $f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i) = \nabla f(\vec{r}_i)\cdot\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$. Thus

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong \nabla f(\vec{r}_i)\cdot\Delta\vec{r}_i = \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

Thus:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \cong \sum_{i=0}^{n-1}\left(f(\vec{r}_{i+1}) - f(\vec{r}_i)\right) =$$

$$= f(\vec{r}_1) - f(\vec{r}_0) + f(\vec{r}_2) - f(\vec{r}_1) + f(\vec{r}_3) - f(\vec{r}_2) + \ldots + f(\vec{r}_n) - f(\vec{r}_{n-1}) =$$

$$= f(\vec{r}_n) - f(\vec{r}_0) = f(Q) - f(P)$$

Thus

$$\int_C \vec{F}\, d\vec{r} = f(Q) - f(P).$$

This is not a precise proof! Only justification.

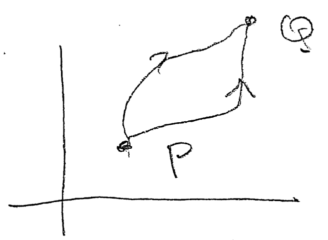
<u>Th</u>: A continuous vector field $\vec{F}$ defined in an open region of the xy-plane is path-independent if and only if $\vec{F}$ is a gradient field; that is:

$$\vec{F} = \nabla f$$

for some function f. In that case, for any piecewise smooth path C from P to Q we have:

$$\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(Q) - f(P).$$

The Fundamental Theorem for Line Integrals.



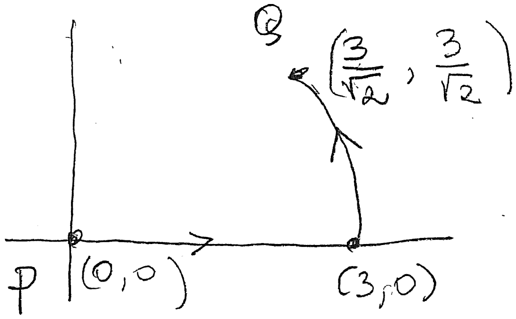
$$\int_P^Q \vec{F}\, d\vec{r} = \int_P^Q \nabla f\, d\vec{r} = f(Q) - f(P).$$

Remember the FTC from Calc I?

$$\int_a^b f'(t)\, dt = f(b) - f(a)$$

<u>Def</u>: If $\vec{F} = \nabla f$, then f is called <u>a potential function of $\vec{F}$</u>.

Ex: Let $\vec{F}(x,y) = x\vec{i} + y\vec{j}$, C be the path:



Find $\int_C \vec{F}\,d\vec{r}$.

We could parametrize both pieces: the segment and a piece of the circle of radius 3. But do we have to?

If $\vec{F} = \nabla f$, then $\int_C \vec{F} = f(Q) - f(P)$.

Is $\vec{F}$ a gradient field?

$$\vec{F}(x,y) = x\vec{i} + y\vec{j} \overset{?}{=} f_x\vec{i} + f_y\vec{j}$$

$f_x = x$, $f_y = y$. Possible.

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

(Or $f(x,y) + C$ for any constant C.)

Hence:

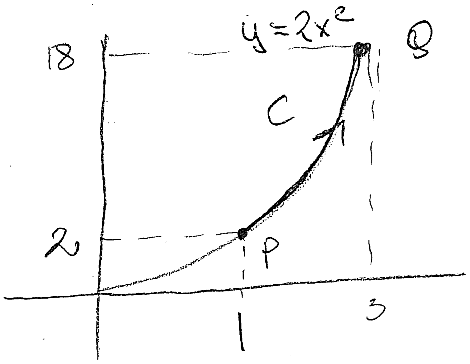
$$\int_C \vec{F}\,d\vec{r} = \int_C \nabla f\,d\vec{r} = f(Q) - f(P) =$$

$$= \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 - 0 = \underline{\underline{\frac{9}{2}}}.$$

Ex: $\vec{F}(x,y) = y\sin(xy)\,\vec{i} + x\sin(xy)\,\vec{j}$, C

is the piece of $y = 2x^2$ from $(1,2)$ to $(3,18)$.

Find $\int_C \vec{F}\,d\vec{r}$.



We could parametrize:

$x(t) = t$

$y(t) = 2t^2$, $\quad t$ in $[1,3]$.

But maybe we can use the FTFLI ?

? $f(x,y)$. $\nabla f = \vec{F}$

$f_x = y\sin(xy)$, $f_y = x\sin(xy)$

$\int$ ↓ $f(x,y) = -y \cdot \frac{1}{y}\cos(xy) + C(y) = -\cos(xy) + C(y)$

$f_y = x\sin(xy) + C'(y)$. Yes. $C(y) \equiv 0$.

The potential function $\underline{f(x,y) = -\cos(xy)}$.

Thus:

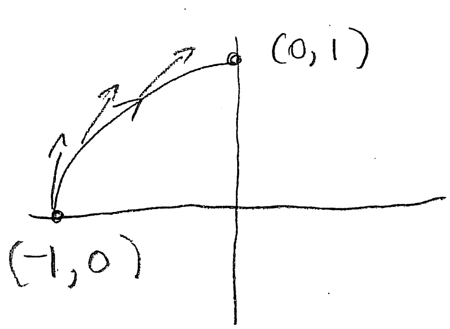
$$\int_C \vec{F}\,d\vec{r} = \int_P^Q \nabla f\,d\vec{r} = f(Q) - f(P) = -\cos(3\cdot 18) + \cos(1\cdot 2) =$$

$$\cong 0.4131\ldots$$

Ex: Suppose a particle subject to a force $\vec{F}(x,y) = y\vec{i} - x\vec{j}$ moves cw along the arc of the unit circle from $(-1,0)$ to $(0,1)$.

(a) Find the work done by $\vec{F}$. Explain the sign of your answer.

(b) Is $\vec{F}$ path-independent? Explain.

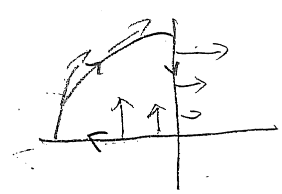


$y\vec{i} - x\vec{j} \perp x\vec{i} + y\vec{j}$

Pointing cw.

(b) Is $\vec{F}$ p-i? Is it a gradient field? $\frac{\partial F_1}{\partial y} = 1$, $\frac{\partial F_2}{\partial x} = -1$. No.

$\vec{F}$ is not p-i. Also if you consider the closed path $\tilde{C}$:



$\int_{\tilde{C}} \vec{F} > 0$. So not p-i. Yet another reason.

(a) We cannot apply the FTC. Hence we have to parametrize:

$\vec{r}(t) = -\cos(t)\vec{i} + \sin(t)\vec{j}$, t in $[0, \frac{\pi}{2}]$

$\vec{r}\,'(t) = \sin(t)\vec{i} + \cos(t)\vec{j}$

$\vec{F}(\vec{r}(t)) = -\sin(t)\vec{i} + \cos(t)\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sin^2(t) + \cos^2(t) = 1$

$$\int_C \vec{F} = \int_0^{\frac{\pi}{2}} 1\,dt = \frac{\pi}{2}.$$