MTH 435 F14 Homework 9 - Solutions

A solution to #2 p.7 is given at the end.

#6 p.8: Consider the family $\mathcal{C}$ of all open sets P contained in Y:

$$\mathcal{C} = \{P : P \subseteq Y, P \text{ open}\}.$$

Denote $U = \bigcup_{P \in \mathcal{C}} P$. Since $\operatorname{int} Y$ is open and $\operatorname{int} Y \subseteq Y$, $\operatorname{int} Y \in \mathcal{C}$.

Thus,

$$\operatorname{int} Y \subseteq U. \quad (1)$$

To prove that

$$U \subseteq \operatorname{int} Y \quad (2)$$

note that U is open as U is a union of open sets and $U \subseteq Y$ as U is a union of subsets of Y. Take $x \in U$. Since U is open, there exists an open ball $B(x,r) \subseteq U$. Since $U \subseteq Y$, $B(x,r) \subseteq Y$. Thus, $x \in \operatorname{int} Y$ and (2) is proved. (1) and (2) give $U = \operatorname{int} Y$ which needed to be shown.

#7 p.8: Let $\mathcal{F}$ be the family of all closed sets containing Y:

$$\mathcal{F} = \{F : F \supseteq Y, F \text{ closed}\}.$$

Let $V = \bigcap_{F \in \mathcal{F}} F$. We want to show that:

$$V = \overline{Y}. \quad (3)$$

Since $\overline{Y}$ is closed and $\overline{Y} \supseteq Y$:

$$V \subseteq \overline{Y} \quad (4)$$

It suffices to show that

$$\overline{Y} \subseteq V, \quad (5)$$

Let $x \in \overline{Y}$. Then x is an adherent point to Y. Since $Y \subseteq V$, any x is an adherent point to V. Since V is closed as an intersection of closed sets, $x \in V$. Thus (5). Therefore $V = \overline{Y}$.

#10 p 8 :

(a) x is a limit point of S if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S$ for $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} x_n = x$.

Proof: "$\Rightarrow$" Assume x is a limit point of S. For every $n = 1, 2, \ldots$ take $B(x, \frac{1}{n})$. $B(x, \frac{1}{n})$ contains points of S different from x. Choose one of them and denote x_n. We have a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$, $x_n \neq x$ for $n \in \mathbb{N}$ and $d(x_n, x) < \frac{1}{n}$. Thus, $\lim_{n\to+\infty} x_n = x$.

"$\Leftarrow$" Assume there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$, $x_n \neq x$, for $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = x$. Take an open ball $B(x, r)$. We have to show that $B(x,r)$ contains infinitely many elements of S. Since $\lim_{n\to+\infty} x_n = x$, for some N, $x_n \in B(x,r)$ for all $n \geq N$. Suppose that the set $\{x_N, x_{N+1}, \ldots, x_n, \ldots\}$ is finite. (Remember elements of in a sequence may repeat.). Then the set $\{d(x_N, x), \ldots, d(x_n, x), \ldots\}$ is a finite set of positive numbers (positive as $x_n \neq x$). Take $r_0 = \min\{d(x_N, x), \ldots, d(x_n, x), \ldots\}$. Then $r_0 > 0$. Also, $x_n \notin B(x, r_0)$ for all $n \geq N$ as $d(x_n, x) \geq r_0$ for $n \geq N$. Contradiction as $x_n \to x$ as $n \to +\infty$.

(b) Let S_L be the set of limit points of S. We want to show that S_L is closed. Let s_0 be a point adherent to S_L. Take any ball $B(s_0, r)$. Since s_0 is adherent to S_L, there exists a point $s_1 \in B(s_0, r)$ such that $s_1 \in S_L$. Since $B(s_0, r)$ is open, there exists $B(s_1, p) \subseteq B(s_0, r)$. Since $s_1 \in S_L$, $B(s_1, p)$ contains infinitely many points of S. So does $B(s_0, r)$. As r was arbitrary $s_0 \in S_L$. S_L contains all its adherent points, so S_L is closed.

#11 p.8 : Denote by S_L the set of all limit points of S, by S_I the set of all isolated points of S. By the definition of an isolated point if $s_0 \in S_I$ then there exists $r > 0$ such that $B(s_0, r) \cap S = \{s_0\}$. Thus, s_0 is not a limit point. Thus, $S_L \cap S_I = \emptyset$. We have to prove that

$$\overline{S} = S_L \cup S_I \qquad (6)$$

We shall prove two inclusions

$$\overline{S} \supseteq S_L \cup S_I \qquad (7)$$

and

$$\overline{S} \subseteq S_L \cup S_I . \qquad (8)$$

To prove (7), take $s_0 \in S_L \cup S_I$. Then $s_0 \in S_I$ or $s_0 \in S_L$. If $s_0 \in S_I$, then $s_0 \in S$. Since $S \subseteq \overline{S}$, $s_0 \in \overline{S}$ and (7) holds. If $s_0 \in S_L$, in every ball $B(s_0, r)$ there are infinitely many points of S. Thus, $B(s_0, r) \cap S \neq \emptyset$ which gives that s_0 is adherent point of S and therefore $s_0 \in \overline{S}$. (7) is proved.

To prove (8), take $s_0 \in \overline{S}$. Suppose $s_0 \notin S$. Since s_0 is adherent to S, for every $n = 1, 2, \dots$, there exists $s_n \in S$ such that $s_n \in B(s_0, \frac{1}{n})$. Then $\lim_{n \to +\infty} s_n = s_0$, $s_n \in S$, $s_n \neq s_0$ for $n \in \mathbb{N}$. By Problem 10, $s_0 \in S_L$ and (8) holds. Suppose $s_0 \in S$. If $s_0 \in S_I$, (8) holds. If $s_0 \notin S_I$, for every $r > 0$, $B(s_0, r) \cap S$ contains other points of S besides s_0. In particular, for every $n = 1, 2, \dots$, there exists $s_n \in S$, $s_n \neq s_0$, $s_n \in B(x_0, \frac{1}{n})$. Since $\lim_{n \to +\infty} s_n = s_0$, $s_0 \in S_L$ by Problem 10.

#14 p. 19 :

(a) We will show that E is open if and only if $E \cap \partial E = \emptyset$.

"$\Rightarrow$" Assume E is open. Take $e \in E$. Since E is open there exists a ball $B(e,r)$ such that $B(e,r) \subseteq E$. Thus, $B(e,r) \cap (X \setminus E) = \emptyset$ which implies that e is not adherent to $X \setminus E$ and thus $e \notin \overline{X - E}$. The latter gives $e \notin \partial E$. Therefore, $E \cap \partial E = \emptyset$.

"$\Leftarrow$" Assume $E \cap \partial E = \emptyset$. Let $e \in E$. Then $e \notin \partial E$. Since $e \in E$, $e \in \overline{E}$. Since $e \notin \partial E = \overline{E} \cap \overline{(X \setminus E)}$, $e \notin \overline{X \setminus E}$. Since e is not adherent to $X \setminus E$, there exists a ball $B(e,r)$ such that $B(e,r) \cap (X \setminus E) = \emptyset$. Thus $B(e,r) \subseteq E$. Since $e \in E$ was arbitrary, E is open.

(b) We will show that E is closed if and only if $\partial E \subseteq E$.

"$\Rightarrow$" If E is closed, $\overline{E} \subseteq E$. Thus $\partial E \subseteq E$ as $\partial E \subseteq \overline{E}$.

"$\Leftarrow$" Assume $\partial E \subseteq E$. We have to show that E is closed or equivalently that $X \setminus E$ is open. By the definition of the boundary $\partial E = \partial(X \setminus E)$. Since $\partial E \subseteq E$, $\partial E \cap (X \setminus E) = \emptyset$. Thus, $\partial(X \setminus E) \cap (X \setminus E) = \emptyset$. By (a) applied to $X \setminus E$, $X \setminus E$ is open. Thus, E is closed.

#2 p. 7 : Let X be an arbitrary nonempty set. The "discrete" metric in X, $d : X \times X \to \mathbb{R}$ is defined as:

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

Clearly, $d(x,y) = d(y,x)$, $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$. Let's prove that d satisfies the Triangle Inequality. Take $x, y, z \in X$. We want to show that

$$d(x,y) \leq d(x,z) + d(y,z). \quad (9)$$

The only only way that (9) may be violated is when $d(x,y) = 1$ and $d(x,z) = d(y,z) = 0$. The latter equalities imply $x = z$, $y = z$, which gives $x = y$ and $d(x,y) = 0$. Thus, (9) always holds. Note, for every $x \in X$, $r > 0$:

$$B(x,r) = \begin{cases} \{x\} & \text{if } r < 1 \\ X & \text{if } r \geq 1. \end{cases}$$

Take any subset $A \subseteq X$. Let $a \in A$. We have

$$B(a, \tfrac{1}{2}) = \{a\} \subseteq A.$$

Hence, A is open. We prove similarly that $X \setminus A$ is open. Therefore, every subset of the discrete metric space (X,d) is both closed and open.