MTH435 F14 Homework 8 – Solutions

1) Since f and g are differentiable at x_0:

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0),\quad \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=g'(x_0).\quad (1)$$

Hence

$$(f+g)'(x_0)=\lim_{x\to x_0}\frac{(f(x)+g(x))-(f(x_0)+g(x_0))}{x-x_0}=\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right)$$

By (1) and Th 19.1 the latter limit exists and is equal to $f'(x_0)+g'(x_0)$. Thus, 27.1(b) is proved.

2) Let $f(x)=c$ be a constant function in a nbhd of x_0. Then

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{c-c}{x-x_0}=\lim_{x\to x_0}0=0.$$

3) f is differentiable at $x=0$ provided the following limit exists:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

If the limit exists it is equal to $f'(0)$. Let's examine the limit:

$$f'(0)=\lim_{x\to 0}\frac{xg(x)-0\cdot g(0)}{x}=\lim_{x\to 0}g(x)$$

As g is continuous at $x=0$, $\lim_{x\to 0}g(x)=g(0)$. Hence, f is differentiable at 0 and $f'(0)=g(0)$.

4) We shall prove that f is differentiable at 0 with $f'(0)=0$. Let $M>0$ be a constant such that

$$|g(x)|\le M \quad \text{for } x\in[-1,1].$$

Such M exists as g is bounded in $[-1,1]$. To prove $f'(0)=0$ we have to show that

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0.\quad (2)$$

We have $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2g(x)}{x}=\lim_{x\to 0}xg(x)$.

To prove that $\lim_{x\to 0}xg(x)=0$, take $\varepsilon>0$. Let $\delta=\frac{\varepsilon}{M}$. Then $\delta>0$ and

$$|xg(x)|=|x||g(x)|\le M|x|<\varepsilon \quad \text{provided } |x|<\delta.$$

Thus, $\lim_{x\to 0}xg(x)=0$ and (2) is proved.

5) Suppose f has more than one fixed points. Then, there exist $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, such that $f(x_1) = x_1$, $f(x_2) = x_2$. Since f is cont. and diff in $\mathbb{R}$, f satisfies the assumptions of the MVT in $[x_1, x_2]$. By the MVT, there exists $c \in [x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{x_2 - x_1}{x_2 - x_1} = 1$ while $f'(c) < 1$. Contradiction. Hence f has at most one fixed point.

6) To prove that $f(x) = |x|$ is cont. at 0, we have to show that

$$\lim_{x \to 0} |x| = 0. \qquad (3)$$

The prove of (3) is trivial from the definition of the limit. Indeed, take $\varepsilon > 0$. Take $\delta = \varepsilon$. Then $|x| < \varepsilon$ provided $|x| < \delta$.

f is not differentiable at 0 as the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} \qquad (4)$$

does not exist. We have:

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Thus, $\lim_{x \to 0+} \frac{|x|}{x} = 1$, $\lim_{x \to 0-} \frac{|x|}{x} = -1$: Since one-sided limits are different the limit in (4) does not exist.

7) Take an abitrary point $x_0 \in (a, b)$. f is differentiable at x_0 and:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad (5)$$

Since f is increasing, $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$ for $x \in (a, b)$. Thus by H5#6, $f'(x_0) \geq 0$.

8) Let $M>0$ be a constant such that

$$|f'(x)| \leq M \quad \text{for all } x \in \mathbb{R}. \quad (6)$$

Take arbitrary $x, y \in \mathbb{R}$, $x \neq y$. f satisfies the assumptions of the MVT on $[x,y]$ or $[y,x]$ (depending which of the two x,y is larger). By the MVT, there exists $c \in (x,y)$ or $c \in (y,x)$ such that

$$f'(c) = \frac{f(y)-f(x)}{y-x}. \quad (7)$$

(7) and (6) imply:

$$\left| \frac{f(y)-f(x)}{y-x} \right| = |f'(c)| \leq M$$

Which in turn gives

$$|f(y)-f(x)| \leq M|y-x| \quad (8)$$

for all $x, y \in \mathbb{R}$, $x \neq y$. (8) holds trivially if $x=y$. Thus (8) holds for any $x, y \in \mathbb{R}$.

(8) implies that f is uniformly continuous in $\mathbb{R}$. Indeed, take any $\varepsilon>0$. Let $\delta = \frac{\varepsilon}{M}$. Then $\delta>0$ and for all $x, y \in \mathbb{R}$:

$$|f(x)-f(y)| < \varepsilon \text{ provided } |x-y|<\delta.$$

Incidentally, a function f which for some constant M satisfies the following condition on an interval I:

$$|f(y)-f(x)| \leq M|x-y| \quad \text{for all } x,y \in I$$

is called a Lipschitzian function on I with a Lipschitz constant M. Lipschitzian functions have many interesting properties that uniformly continuous functions do not have.