1) To prove that $f(x)=x^2$ is not uniformly continuous on $I=[0,+\infty)$ we have to show that:

$$\underset{\varepsilon_0>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{x_\delta,y_\delta\in I}{\exists}\ \left(|x_\delta-y_\delta|<\delta \wedge |f(x_\delta)-f(y_\delta)|\geq\varepsilon_0\right). \quad (1)$$

Take $\varepsilon_0=1$. Let $\delta>0$ be given. Take $x_\delta=\frac{1}{\delta}$, $y_\delta=\frac{1}{\delta}+\frac{\delta}{2}$.

Then $x_\delta, y_\delta\in I$ and:

$$|x_\delta-y_\delta|=\frac{\delta}{2}<\delta.$$

Also:

$$|f(x_\delta)-f(y_\delta)|=|x_\delta^2-y_\delta^2|=|x_\delta-y_\delta|\cdot|x_\delta+y_\delta|=\frac{\delta}{2}\cdot\left(\frac{2}{\delta}+\frac{\delta}{2}\right)\geq$$

$$\geq\frac{\delta}{2}\cdot\frac{2}{\delta}=1$$

Thus, (1).

2) Take $\varepsilon>0$. As f and g are uniformly continuous on I, there exist $\delta_1>0$, $\delta_2>0$ such that

$$|f(x)-f(y)|<\frac{\varepsilon}{2} \quad \text{provided } |x-y|<\delta_1,\ x,y\in I$$

$$|g(x)-g(y)|<\frac{\varepsilon}{2} \quad \text{provided } |x-y|<\delta_2,\ x,y\in I$$

Let $\delta=\min\{\delta_1,\delta_2\}$. Then $\delta>0$ and

$$|f(x)-f(y)|<\frac{\varepsilon}{2},\ |g(x)-g(y)|<\frac{\varepsilon}{2} \quad \text{provided } |x-y|<\delta,\ x,y\in I. \quad (2)$$

(2) implies:

$$|(f(x)+g(x))-(f(y)+g(y))|\leq|f(x)-f(y)|+|g(x)-g(y)|<\varepsilon$$

provided $|x-y|<\delta$, $x,y\in I$.

Therefore, $f+g$ is uniformly continuous on I.

3) Take $f(x)=x$, $g(x)=x$, $I=[0,+\infty)$. The product $(f\cdot g)(x)=x^2$ is not uniformly continuous on I by Problem 1. $f(x)$ (and therefore $g(x)$) is uniformly continuous on I. Indeed, for every $\varepsilon>0$ take $\delta=\varepsilon$. We have for every $x,y\in I$:

$$|f(x)-f(y)|=|x-y|<\varepsilon \quad \text{provided } |x-y|<\delta.$$

4) It sufficies to prove that the function

$$h(x)=1+x-\cos x-C$$

is 0 at some $x_0\in\mathbb{R}$. Note that h is continuous on $(-\infty,+\infty)$ as

$1, c, \cos x, x$ are all continuous. Observe:

$$h(x) = 1 + x - \cos x - c \leq x - c + 2 < 0 \text{ for } x < c - 2$$
$$h(x) = 1 + x - \cos x - c \geq x - c > 0 \quad \text{for } x > c.$$

The inequalities follow as $|\cos x| \leq 1$. Take $a < c-2$, $b > c$. Then $f(a) < 0$, $f(b) > 0$. Thus, by Corollary 24.1, there exists $x_0 \in (a, b)$ such that $f(x_0) = 0$. (Cor 24.1 is a corollary from the IVT.)

5) Suppose the contrary; that is, suppose that for some $x_1, x_2 \in \mathbb{R}$ $f(x_1) \neq f(x_2)$. Without any loss of generality we may assume that $f(x_1) < f(x_2)$. Let $q \in \mathbb{Q}$ be such that $f(x_1) < q < f(x_2)$. Such q exists by the density of rationals. By the IVT or more precisely by Cor. 24.1, there exists c in (x_1, x_2) or (x_2, x_1) such that $f(c) = q$. Contradiction as f takes only irrational values. Thus, f must be constant.

6) Consider the function $h(x) = x - f(x)$. As $f: [0,1] \to [0,1]$, $h(0) \leq 0$. If $h(0) = 0$, $x = 0$ is a fixed point. Suppose then $h(0) < 0$. As $f(1) \leq 1$, $h(1) = 1 - f(1) \geq 0$. If $h(1) = 0$, then $1 - f(1) = 0$, $f(1) = 1$, and $x = 1$ is a fixed point. Suppose then $h(1) > 0$. h is continuous in $[0, 1]$, $h(0) < 0$, and $h(1) > 0$. By the IVT, there exists $c \in (0, 1)$ such that $h(c) = c - f(c) = 0$. Then $f(c) = c$ and c is a fixed point.

7) Consider the function $h(x) = f(x) - g(x)$. h is continuous in $(-\infty, +\infty)$ and $h(x) = 0$ whenever $x \in \mathbb{Q}$. Take any $y \in \mathbb{R}$. Take a sequence $q_n \in \mathbb{Q}$, $n = 1, 2, \dots$, $\lim\limits_{n \to +\infty} q_n = y$. By Heine's characterization of continuity, $\lim\limits_{n \to +\infty} h(q_n) = h(y)$. Since $h(q_n) = 0$ for $n \in \mathbb{N}$, $h(y) = 0$. Thus, h is constant and equal to 0 in $(-\infty, +\infty)$ which gives $f(x) = g(x)$ for $x \in (-\infty, +\infty)$. Note: the existence of the sequence $\{q_n\}_{n=1}^{\infty}$ follows from Pr. 3 Homework 5.

8) Suppose that there exists an $a_0 \in \mathbb{R}$ such that f attains a_0 only finitely many times; that is, there is a finite number of values of x, say $x_1, \ldots, x_m$ for some $m \in \mathbb{N}$, such that $f(x_i) = a$ for $i = 1, \ldots, m$. Let b be such that $b \geq \max\{x_1, \ldots, x_m\}$.

Then the closed bounded interval $[0, b]$ contains all points where f takes value a. In other words:

$$f(x) \neq a \quad \text{for all } x > b. \qquad (3)$$

(3) implies that

$$f(x) > a \text{ for all } x > b \quad \text{or} \quad f(x) < a \text{ for all } x > b. \quad (4)$$

Indeed, if at some point $\bar{x}, \bar{\bar{x}} > b$ we have $f(\bar{x}) > a$ and $f(\bar{\bar{x}}) < a$ then by the IVT the exists a point c between $\bar{x}, \bar{\bar{x}}$ such that $f(c) = a$. That contradicts (3). Thus, (4) holds.

Suppose
$$f(x) > a \quad \text{for all } x > b. \qquad (5)$$

By the Boundedness Theorem, for some $M > 0$ we have

$$-M \leq f(x) \leq M \qquad \text{for all } x \in [0, b]. \quad (6)$$

Take $K = \min\{a, -M\}$. Then by (5) and (6):

$$f(x) \geq K \quad \text{for all } x \in [0, +\infty).$$

The latter contradicts the assumption that f is not bounded below. Thus, (5) cannot hold. Suppose the second part of (4) holds; that is:

$$f(x) < a \quad \text{for all } x > b. \quad (7)$$

Take $K = \max\{M, a\}$. By (6) and (7):

$$f(x) \leq K \quad \text{for all } x \in [0, +\infty).$$

Again a contradiction as f is not bounded above. Thus, either possibility in (4) leads to a contradiction. Therefore, the supposition that f attains a finitely many times leads to a contradiction and the theorem of Problem 8 is proved. ∎