2) We prove part (A). The proof of (B) is similar. Let $c \in \mathbb{R}$ be such that $f(c) > 0$. Take $\bar{\varepsilon} = \frac{f(c)}{2}$. Then $\bar{\varepsilon} > 0$. Since f is continuous at c in I there exists a $\bar{\delta} > 0$ such that:

$$|f(x) - f(c)| < \bar{\varepsilon} \quad \text{provided} \quad |x-c| < \bar{\delta}, \ x \in I.$$

Since x is an interior point of I, for some $\delta > 0$, $(x-\delta, x+\delta) \subseteq I$. Without any loss of generality, we can assume $\bar{\delta} \le \delta$. (If not, we replace $\bar{\delta}$ with $\min\{\bar{\delta}, \delta\}$.) Thus we have:

$$|f(x) - f(c)| < \bar{\varepsilon} \quad \text{provided} \quad |x-c| < \bar{\delta}. \qquad (1)$$

(1) implies, in particular:

$$f(c) - \bar{\varepsilon} < f(x) \quad \text{for} \quad x \in (c-\bar{\delta}, c+\bar{\delta}).$$

Since $\bar{\varepsilon} = \frac{f(c)}{2}$, we obtain:

$$f(x) > \frac{f(c)}{2} > 0 \quad \text{for } x \in (c-\bar{\delta}, c+\bar{\delta}).$$

Therefore, (A) holds with $\varepsilon_0 = \bar{\delta}$.

If $f(c) = 0$ we cannot conclude anything about the sign of f in a nbhd c.

3) Here we can use a direct proof or a proof by contradiction. Let's look at both.

<u>A direct proof</u>: We want to prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Take an arbitrary $\bar{x} \in \mathbb{R}$. By Homework 5, Probem 3, we know that there exists a sequence of rationals convergent to $\bar{x}$: $\{q_n\}_{n=1}^{\infty}$, $q_n \in \mathbb{Q}$ for $n \in \mathbb{N}$, $\lim\limits_{n\to+\infty} q_n = \bar{x}$. Since f is continuous, we obtain by Heine's characterization of the limit that $\lim\limits_{n\to+\infty} f(q_n) = f(\bar{x})$. But $f(q_n) = 0$. Thus, $\lim\limits_{n\to+\infty} f(q_n) = 0 = f(\bar{x})$. Since $\bar{x} \in \mathbb{R}$ was arbitrary, $f(x) = 0$ for all $x \in \mathbb{R}$.

<u>An alternative proof by contradiction</u>: Suppose there exists $c \in \mathbb{R}$ such that $f(c) \neq 0$. Suppose first that $f(c) > 0$. Then by Prop H6.1 there exists a n'hd $(c-\varepsilon_0, c+\varepsilon_0)$ such that $f(x) > 0$ for all $x \in (c-\varepsilon_0, c+\varepsilon_0)$. Contradiction as by the density of rationals $(c-\varepsilon_0, c+\varepsilon_0)$ contains rational numbers. We reason similarly in the case $f(c) < 0$. By Prop H6.1, in some nbhd $(c-\varepsilon_0, c+\varepsilon_0)$, $f(x) < 0$ for all $x \in (c-\varepsilon_0, c+\varepsilon_0)$. A contradiction as the nbhd contains rational numers and $f(q) = 0$ for all $q \in \mathbb{Q}$.

4) $f(x) = \frac{1}{\sqrt{x}}$ is defined in $(0, +\infty)$. Take an $\varepsilon > 0$.
Take a p such that $p > \frac{1}{\varepsilon^2}$. Then:

$$\left| \frac{1}{\sqrt{x}} - 0 \right| = \frac{1}{\sqrt{x}} < \varepsilon \quad \text{whenever} \quad x > p ..$$

Indeed, $x > p$ gives $x > \frac{1}{\varepsilon^2}$ which in turn implies $\frac{1}{\sqrt{x}} < \varepsilon$.
Thus, by Def H6.1, $\lim_{x \to +\infty} \frac{1}{\sqrt{x}} = 0$.

5) We use Def H6.1. Let $\varepsilon > 0$ be given. As $\lim_{x \to +\infty} f(x) = L$, $\lim_{x \to +\infty} g(x) = M$, there exist $p_1 > a$, $p_2 > a$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{for } x \geq p_1 \qquad (2)$$

$$|g(x) - M| < \frac{\varepsilon}{2} \quad \text{for } x \geq p_2. \qquad (3)$$

Take $p = \max\{p_1, p_2\}$. Then for $x \geq p$ (2) and (3) hold. We obtain:

$$|f(x) + g(x) - (L+M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } x \geq p. \quad (4)$$

Since ε was arbitrary, $\lim_{x \to +\infty} (f(x) + g(x)) = L + M$ by (4).

6) <u>Def H6.2</u>: Let f be defined in $(-\infty, a)$ for some $a \in \mathbb{R}$.
Let $L \in \mathbb{R}$. We say that

$$\lim_{x \to -\infty} f(x) = L$$

if

$$\forall_{\varepsilon > 0} \; \exists_{p < a} \; \forall_{x \in (-\infty, a)} \; (x \leq p \Rightarrow |f(x) - L| < \varepsilon).$$