Homework 5 – Solutions to Problems 1, 2, 3, 5, 4 MTH 435, F14

1) Suppose that $L > M$. Take $\varepsilon_0 = \frac{L-M}{2}$. Since $L > M$, $\varepsilon_0 > 0$. Thus, there exist $N_1, N_2 \in \mathbb{N}$ such that

$|a_n - L| < \varepsilon_0$ for $n \geq N_1$

$|b_n - M| < \varepsilon_0$ for $n \geq N_2$

Take $N = \max\{N_0, N_1, N_2\}$. Then for all $n \geq N$ we have:

$$|a_n - L| < \varepsilon_0, \quad |b_n - M| < \varepsilon_0, \quad a_n \leq b_n.$$

In particular, for $n = N$ we obtain:

$$L - \varepsilon_0 < a_N, \quad b_N < M + \varepsilon_0 \quad a_N \leq b_N. \quad (1)$$

Since $\varepsilon_0 = \frac{L-M}{2}$, $L - \varepsilon_0 = \frac{L+M}{2}$, $M + \varepsilon_0 = \frac{L+M}{2}$. (1) gives

$$a_N > \frac{L+M}{2} > b_N, \quad a_N \leq b_N.$$

Contradiction. Thus, $L \leq M$.

2) Take $\{a_n\}_{n=1}^{\infty}$, $a_n = -1$ for all $n \in \mathbb{N}$. Take $\{c_n\}_{n=1}^{\infty}$, $c_n = 1$ for all $n \in \mathbb{N}$. $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are constant sequeces and, therefore, they are Cauchy sequences. Let $\{b_n\}_{n=1}^{\infty}$ be the sequence $b_n = (-1)^n$, $n \in \mathbb{N}$. Then

$$a_n \leq b_n \leq c_n \quad \text{for } n = 1, 2, \ldots$$

$\{b_n\}_{n=1}^{\infty}$ is divergent as proved in class. Thus, $\{b_n\}_{n=1}^{\infty}$ is not Cauchy.

3) Since rationals and irrationals are dense in $\mathbb{R}$ for every $n = 1, 2, \ldots$ there exist $q_n \in \mathbb{Q}$, $p_n \in \mathbb{R} \setminus \mathbb{Q}$ such that:

$$a < q_n < a + \tfrac{1}{n}, \quad a < p_n < a + \tfrac{1}{n}$$

As $\lim_{n \to +\infty} a = \lim_{n \to +\infty} (a + \frac{1}{n}) = a$ the Squeeze Theorem gives $\lim_{n \to +\infty} q_n = a$ and $\lim_{n \to +\infty} p_n = a$.

5) $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$. Indeed, $x^2 \cos \frac{1}{x}$ is defined for all $x \neq 0$. Thus $x^2 \cos \frac{1}{x}$ is defined in a deleted nbhd N_d of 0.

Let $\varepsilon > 0$ be given. Take $\delta = \sqrt{\varepsilon}$. Then for every $x \in N_d$:

$$|x^2 \cos \tfrac{1}{x}| \leq |x^2| = |x|^2 < \varepsilon \text{ provided } |x| < \delta.$$

4) Let $c \in \mathbb{R}$ be given. By Problem 3 there exist sequences $\{q_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ such that $q_n \neq c$, $p_n \neq c$, $q_n \in \mathbb{Q}$, $p_n \in \mathbb{R} \setminus \mathbb{Q}$ for $n \in \mathbb{N}$, $\lim_{n \to +\infty} q_n = \lim_{n \to +\infty} p_n = c$. As $D(q_n) = 1$, $D(p_n) = 0$ for $n \in \mathbb{N}$, $\lim_{n \to +\infty} D(p_n) = 0 \neq \lim_{n \to +\infty} D(q_n) = 1$. Thus, by Heine's characterization $\lim_{x \to c} D(x)$ does not exist.