1) $\lim_{n\to+\infty} \frac{(-1)^n}{\sqrt{n}} = 0$. We prove it using Def 10.4. Let $\varepsilon > 0$ be given.

Let $N_\varepsilon \in \mathbb{N}$ be such that $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then for every $n \geq N_\varepsilon$ we have

$$\left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}} < \varepsilon.$$

∎

2) Since $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ are bounded, there exist $K, M \in \mathbb{R}$ such that

$$|a_n| \leq K, \quad |b_n| \leq M \quad \text{for all } n \in \mathbb{N}. \qquad (*)$$

Using the Triangle Inequality, $(*)$, and Proposition 10.0, we obtain for all $n \in \mathbb{N}$:

$$|a_n + b_n| \leq |a_n| + |b_n| \leq K + M,$$

$$|a_n - b_n| = |a_n + (-b_n)| \leq |a_n| + |b_n| \leq K + M,$$

$$|a_n \cdot b_n| = |a_n| \cdot |b_n| \leq K \cdot M.$$

(By $(*)$, $K \geq 0$ and $M \geq 0$. Also, $|b_n| = |-b_n|$ by Prop 10.0).

Hence, $\{a_n + b_n\}$, $\{a_n - b_n\}$, $\{a_n \cdot b_n\}$ are all bounded as $K+M, K\cdot M \in \mathbb{R}$.

3) We use Def 10.4. Let $\varepsilon > 0$ be given. Since $\lim_{n\to+\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for } n \geq N_1. \qquad (2)$$

Let $N = \max\{N_1, N_0\}$. Then for any $n \geq N$, (2) holds as well as $c_n = a_n$. Thus:

$$|c_n - L| < \varepsilon \text{ for } n \geq N.$$

We conclude $\lim_{n\to+\infty} c_n = L$.

∎

Observe that the latter theorem means that convergence and the limit of a given sequence does not depend on any finite member of its elements. In other words, you can change any finite number of elements of a sequence without affecting its limit or convergence.

4) We use the definition of the limit. Take $\varepsilon > 0$. Let $N_\varepsilon \in \mathbb{N}$ be such that
$$N_\varepsilon > \frac{\sqrt{5}}{\sqrt{\varepsilon}}.$$
Then for every $n \geq N_\varepsilon$, $n^2 > \frac{5}{\varepsilon}$ which gives $\frac{5}{n^2} < \varepsilon$.
We have then:
$$\left|\left(1-\frac{5}{n^2}\right)-1\right| = \frac{5}{n^2} < \varepsilon \quad \text{for } n \geq N_\varepsilon.$$
Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to+\infty}\left(1-\frac{5}{n^2}\right) = 1$. ■

5) Let $K > 0$ be such that:
$$|a_n| \leq K \text{ for } n \in \mathbb{N}. \quad (3)$$
Such K exists as $\{a_n\}_{n=1}^{\infty}$ is bounded. We have $|a_n b_n| = |a_n||b_n|$, $n \in \mathbb{N}$. Thus, by (3)
$$0 \leq |a_n b_n| \leq K|b_n|$$
As $\lim_{n\to+\infty} b_n = 0$ we have $\lim_{n\to+\infty} K|b_n| = 0$ by Th 12.4. By the Squeeze Theorem we obtain:
$$\lim_{n\to+\infty} |a_n b_n| = 0$$
By Prop 12, 2, $\lim_{n\to+\infty}(a_n b_n) = 0$. ■

6) Since
$$s_1 = \sup T, \quad s_2 = \sup V \quad (4)$$
we have
$$\forall_{t\in T}\ t \leq s_1 \quad \text{and} \quad \forall_{v \in V}\ v \leq s_2. \quad (5)$$
Since every element of $s \in S$ is of the form $s = t+v$ for some $t \in T$, $v \in V$, (5) gives for every $s \in S$:
$$s = t + v \leq s_1 + s_2.$$
Thus, $s_1 + s_2$ is an upper bound for S. To show that $s_1 + s_2 = \sup S$, we use Th 9.3. Take $\varepsilon > 0$. By (4) and Th 9.3, there exist $t_\varepsilon \in T$, $v_\varepsilon \in V$ such that:
$$s_1 - \frac{\varepsilon}{2} < t_\varepsilon \leq s_1, \quad s_2 - \frac{\varepsilon}{2} < v_\varepsilon \leq s_2.$$
Thus:
$$s_1 + s_2 - \varepsilon < t_\varepsilon + v_\varepsilon \leq s_1 + s_2.$$
Since $t_\varepsilon + v_\varepsilon \in S$ and ε was arbitrary, $s_1 + s_2 = \sup S$. ■

7) You can find a proof of the Reverse Triangle Inequality in Gordon, page 11.

8) "$\Leftarrow$" Assume s is an upper bound of A and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \in A$ for $n \in \mathbb{N}$ and $\lim_{n\to+\infty} a_n = s$. We prove that $s = \sup A$ using Th 9.3. Part (a) holds as s is an upper bound for A. To prove Part (b), take $\varepsilon > 0$. As $\lim_{n\to+\infty} a_n = s$, there exists $N \in \mathbb{N}$ such that:

$$s - \varepsilon < a_n < s + \varepsilon \quad \text{for} \quad n \geq N.$$

In particular, for $n = N$:

$$s - \varepsilon < a_N \quad \text{and} \quad a_N \in A.$$

By Th 9.3, $s = \sup A$.

"$\Rightarrow$" Assume that $s = \sup A$. We will show that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of elements of A which converges to s. We will use Th 9.3 (b). For every $n \in \mathbb{N}$, $\varepsilon = \frac{1}{n} > 0$. Thus, there exists an element of A, denote it a_n, such that:

$$s - \varepsilon < a_n \leq s < s + \varepsilon.$$

Thus for every $n \in \mathbb{N}$:

$$a_n \in A \quad \text{and} \quad s - \frac{1}{n} < a_n \leq s. \qquad (6)$$

The sequence $\{a_n\}_{n=1}^{\infty}$ converges to s by (6) and the Squeeze Theorem as $\lim_{n\to+\infty}\left(s - \frac{1}{n}\right) = s = \lim_{n\to+\infty} s$. ■