MTH 435 Homework 3 - Solutions F14

1) Proof very similar to Proof of Th 9.3 by obvious modifications and Def 7.3.

2) As $B \subseteq (1,2)$, $b < 2$ for all $b \in B$, so 2 is an upper bound for B. To prove that $2 = \sup B$ we use Th 9.3 and Th 9.1. Let $\varepsilon > 0$ be given. By Th 9.1, there exists a rational number q such that

$$\max\{2-\varepsilon, 1\} < q < 2.$$

Then $q \in B$. As ε was chosen arbitrarily, $s = 2$ satisfies both conditions (a), (b) of Th 9.3. Hence, $2 = \sup B$.

3) Prove very similar to 2) using Th 9.4 and Th 9.2.

4) Since $A \neq \emptyset$, there exists $a \in A$. By Def 7.2 and Def 7.3 we have then:

$$\inf A \leq a \leq \sup A.$$

Thus, $\inf A \leq \sup A$. The equality holds for any singleton, that is, a one-element set. For example, $A = \{0\}$.

5) "$\Rightarrow$" Assume that A has the maximum, $a_0 = \max A$. By Def H3.1, a_0 is an upper bound for A. Since $a_0 \in A$ no number smaller than a_0 can be an upper bound for A. Hence, $a_0 = \sup A$ and $\sup A \in A$.

"$\Leftarrow$" Let $s = \sup A$. Assume that $s \in A$. Since, s is an upper bound we have $a \leq s$ for all $a \in A$. Thus, be Def H3.1, s is the maximum of A.

6) To prove that S in dense, take any $a, b \in \mathbb{R}$, $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ and by Th 9.1, there exists $q \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}.$$

Since $q \in \mathbb{Q}$, $q = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Hence,

$$a < \frac{m\sqrt{2}}{n} < b.$$

Since $\frac{m\sqrt{2}}{n} \in S$, the claim is proved.

7) Since $A \neq \emptyset$, $B \neq \emptyset$ and

$$\forall_{a \in A} \forall_{b \in B} \quad a < b$$

every $b \in B$ is an upper bound for A and every $a \in A$ is a lower bound for B. Thus A is bounded above, B is bounded below. By the Completeness Axiom the supremum of A and the infimum of B exist:

$$\sup A \in \mathbb{R}, \quad \inf B \in \mathbb{R}$$

Since every $b \in B$ is an upper bound for A and $\sup A$ is the least upper bound for A, we have:

$$\forall_{b \in B} \quad \sup A \leq b$$

Thus, $\sup A$ is a lower bound for B and since $\inf B$ is the greatest lower bound, we obtain:

$$\sup A \leq \inf B. \qquad (1)$$

Since $A \cup B = \mathbb{R}$, $\sup A \in A$ or $\sup A \in B$. If $\sup A \in A$, by Problem 5 of this homework $\sup A$ is the largest element of A.
If $\sup A \in B$, $\sup A \geq \inf B$ as $\inf B$ is a lower bound for B.
The latter inequality and (1) give in this case:

$$\sup A = \inf B.$$

Thus, $\inf B \in B$ as $\sup A \in B$. $\inf B \in B$ and $\inf B$ is a lower bound for B, thus, $\inf B$ is the smallest element in B.
We proved that $\sup A \in A$ implies that A has the largest element and $\sup A \in B$ implies that B has the smallest element.

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