Partial Derivatives Primer

If you didn't have MTH 243, here is a primer on partial derivatives. MTH 243 – Multivariable Calculus – talks about functions of more than one variable.

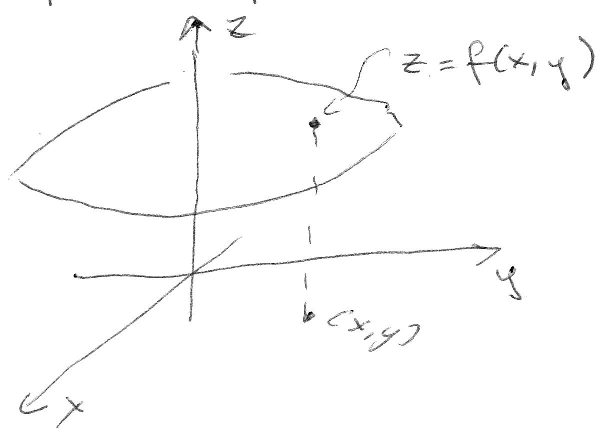
Consider a function of two variables:

$$z = f(x, y)$$

the dependent variable.

independent variables

The graph of such function is a surface in the xyz-space:



If you fix one variable, say $y = b$, and consider $f(x, b)$, $f(x, b)$ is a function of x only. To calculate partial deriatives, say $f_x(x,y)$, you consider one variable fixed, for $f_x(x,y)$ y is fixed, and calculate the derivative with respect to the other variable. More precise notes follow. (Section numbers are from MTH 243 textbook.)

14.1, 14.2 Partial Derivatives

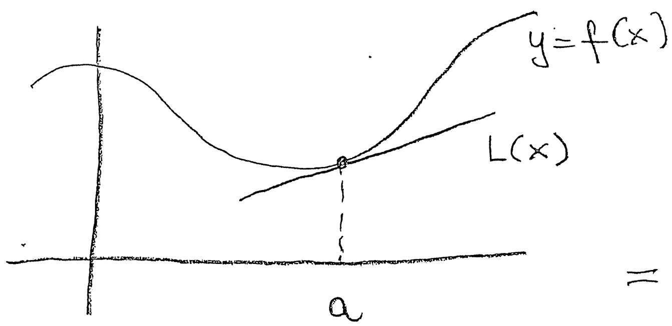
Going back to functions of two variables:

$$z = f(x, y).$$

Today we are going to define and interpret partial derivatives of $f(x, y)$ with respect to x and y.

Recall for a function of one variable:

$$y = f(x)$$



$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = m_{tan}$$

$$= \begin{pmatrix} \text{The rate of change} \\ \text{of } y \text{ at } x = a \end{pmatrix} \frac{\text{units of } y}{\text{unit of } x}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So

$$f(x) \cong f(a) + f'(a)(x-a) \quad \leftarrow \quad (x = a + h)$$

The equation of the tangent line:

$$L(x) = f(a) + f'(a)(x-a)$$

$L(x)$ local linearization of $f(x)$ at $x = a$.

$f(x)$ differentiable at $x = a$ $\Leftrightarrow$ the tangent line exists.

How much of that translates to $f(x, y)$ and how?

The first step : partial derivatives.

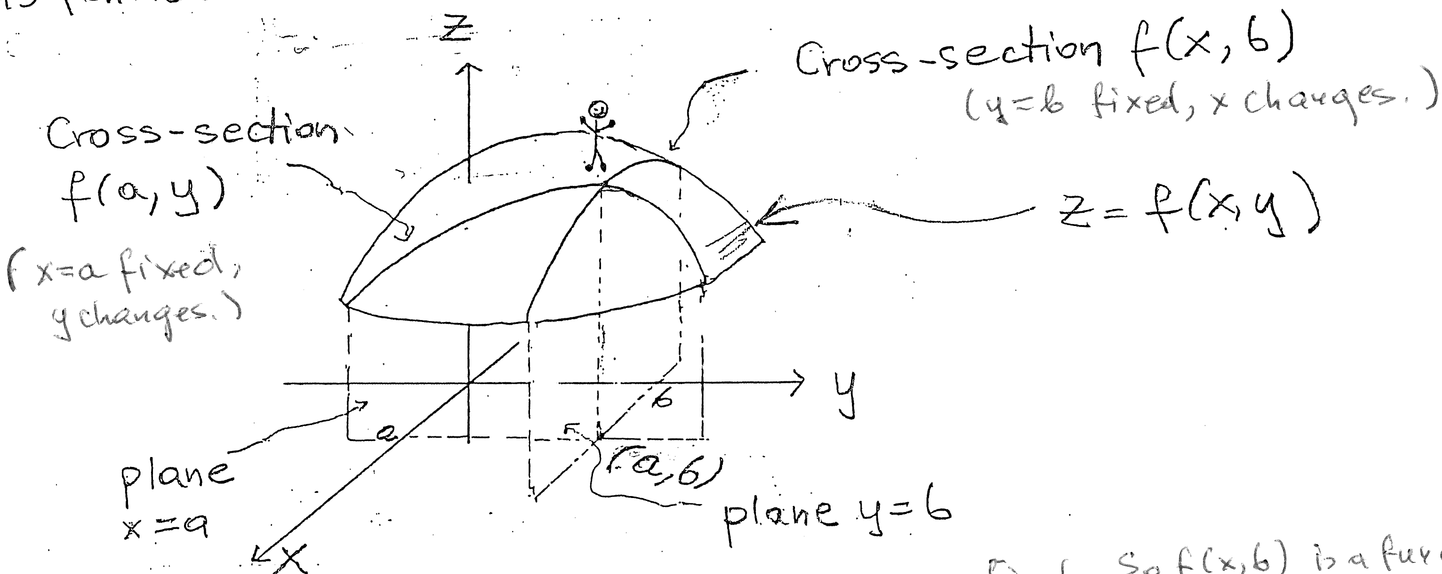
Let a function $z = f(x, y)$ and a point (a, b) in its domain be given. We define :

$f_x(a, b)$ – the partial derivative of $f(x, y)$ with respect to x at (a, b)

and

$f_y(a, b)$ – the partial derivative of $f(x, y)$ with respect to y at (a, b).

as follows.



Fixed. So $f(x, b)$ is a function of x only.

$$f_x(a, b) = \frac{d}{dx}\Big|_{x=a} [\, f(x, b) \,]$$

Fixed

$$f_y(a, b) = \frac{d}{dy}\Big|_{y=b} [\, f(a, y) \,]$$

In other words

$f_x(a, b)$ = the slope of $f(x, b)$ at $x = a$.

$f_y(a, b)$ = the slope of $f(a, y)$ at $y = b$.

So to calculate

$$f_x(a, b)$$

we fix $y = b$, consider the function $f(x, b)$ which is a function of one variable x, and calculate the ordinary derivative of $f(x, b)$ with respect to x.

Similarly, to calculate

$$f_y(a, b)$$

we fix $x = a$, consider $f(a, y)$ etc.

Ex: Let $f(x, y) = x^2 + y^3$. Find $f_x(2, 1)$, $f_x(1, 3)$.

To find $f_x(2, 1)$, we fix $y = 1$ and consider the cross-section

$$f(x, 1) = x^2 + 1$$

Now we take the ordinary derivative

$$\frac{d}{dx}[f(x, 1)] = \frac{d}{dx}[x^2 + 1] = 2x ;$$

And evaluate at $x = 2$: $f_x(2, 1) = 2x \big|_{x=2} = 4$

For $f_x(1, 3)$, we take

$$f(x, 3) = x^2 + 27$$

$$\frac{d}{dx}\Big|_{x=1}[x^2 + 27] = 2x\big|_{x=1} = 2.$$

Clearly calculating $f_x(a,b)$, $f_y(a,b)$ for each given point separately is silly. For a given

$$f(x,y)$$

we should calculate partial derivatives functions:

$$f_x(x,y)\ ,\quad f_y(x,y)$$

and then evaluate at any point (a,b) we want. How to do this?

Ex: Let $f(x,y)=x^2+y^3$. Find $f_x(x,y)$ and $f_y(x,y)$. Find $f_y(2,1)$.

To find $f_x(x,y)$, we take $f(x,y)=x^2+y^3$ and assume that y is fixed so y is a constant. Assuming that y is a constant, we take the derivative of $f(x,y)=x^2+y^3$ with respect to x:

$$f_x(x,y) = (x^2+y^3)'_x = 2x$$

To find $f_y(x,y)$, we assume that x is a constant and differentiate with respect to y:

$$f_y(x,y) = (x^2+y^3)'_y = 3y^2.$$

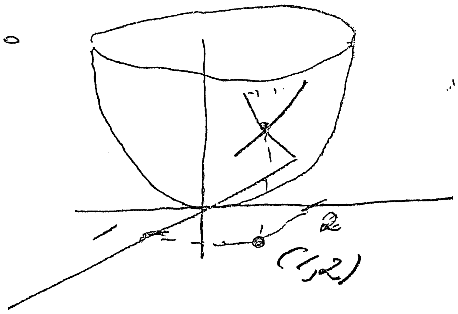
So $f_y(2,1)=3$. Easy!

Ex: Let $f(x,y) = x^2 + y^2$. Find $f_x(1,2)$, $f_y(1,2)$.

$f_x(x,y) = 2x$, $f_y(x,y) = 2y$

$f_x(1,2) = 2$, $f_y(1,2) = 4$.

So



For the paraboloid at (1,2) the slope in the x direction is 2; the slope in the y direction is 4.

Leibnitz Notation:

$z = f(x,y)$, $f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x,y)]$,

$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x,y)]$

$f_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)}$, $f_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)}$

Ex: $z = x^2 y$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^2 y] = 2yx$, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^2 y] = x^2$.

For a given $z = f(x, y)$, find partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

<u>Ex:</u> $z = \sin(xy^3)$.

$$\frac{\partial z}{\partial x} = \cos(xy^3) \cdot y^3$$

$$\frac{\partial z}{\partial y} = \cos(xy^3) \cdot 3xy^2$$

<u>Ex:</u> $z = x^2 e^{x \cdot y}$

$$\frac{\partial z}{\partial x} = 2x\, e^{xy} + x^2 y e^{xy}$$

$$\frac{\partial z}{\partial y} = x^2 \cdot x\, e^{xy} = x^3 e^{xy}$$

<u>Practice!</u>

Of course partial derivatives, $f_x(a, b)$ $f_y(a, b)$ are rates of change of $f(x, y)$ at (a, b) in the x direction any the y direction.

Let $z = f(x,y)$. We defined partial derivatives functions denoted:

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x,y)] = z_x = \frac{\partial f}{\partial x} = f_x$$

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x,y)] = z_y = \frac{\partial f}{\partial y} = f_y$$

Computing partials algebraically is easy: we consider one variable to be a constant and differentiate with respect to the other.

Ex: Let $f(x,y) = x^3 + 3x^2y + y^2$. Find $f_x(x,y)$ and $f_y(x,y)$.

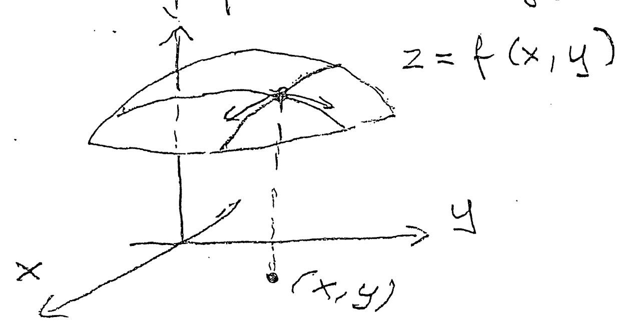
$$f_x(x,y) = \frac{\partial}{\partial x}[x^3 + 3x^2y + y^2] = 3x^2 + 6xy$$

y constant, take the derivative in x.

$$f_y(x,y) = \frac{\partial}{\partial y}[x^3 + 3x^2y + y^2] = 3x^2 + 2y$$

x constant, take the derivative in y.

Geometrically, we found slopes of $z = f(x,y)$ in the x and y directions at any point (x,y):



14.7 Second Partials

Given $z = f(x,y)$ each partial derivative

$$\frac{\partial z}{\partial x} = f_x(x,y) \quad , \quad \frac{\partial z}{\partial y} = f_y(x,y)$$

is again a function of (x,y). So we can take their partial derivatives. Partial derivatives of partial derivatives are called second partial derivatives. We denote them:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[f_x(x,y)]$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}[f_x(x,y)]$$

first second   second first

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[f_y(x,y)]$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}[f_y(x,y)].$$

Four partials. The partials:

$$f_{xy} \, , \, f_{yx}$$

are called mixed partials.

Ex: $z = f(x,y)$, $f(x,y) = x^3 + x^2y^2 - y^4$.

Find second partials.

$f_x = 3x^2 + 2xy^2$, $f_y = 2x^2y - 4y^3$

$f_{xx} = 6x + 2y^2$; $f_{xy} = 4xy$

$f_{yy} = 2x^2 - 12y^2$, $f_{yx} = 4xy$

$f_{xy} = f_{yx}$

Is it always so?

Th: If $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Second partials important, among others, in the context of

- Taylor approximations of order 2
- Second Derivative Test for Local Extrema

Remark: We are not going to talk much about conditions for differentiability of a function $z = f(x,y)$ but let us state at least the following:

Th: If $f_x(x,y)$, $f_y(x,y)$ exist and are continuous in a disk centered at (a,b), then $f(x,y)$ is differentiable at (a,b).