1) From Fatou's Lemma we deduce

\[ \sum f \leq \liminf \sum f_n. \quad (1) \]

(\(\sum f\) denotes \(\int f\)). The condition \(f_n \leq f\) implies via Th 21.1

\[ \sum f_n \leq \sum f \quad \text{for all } n = 1, 2, \ldots \]

Hence,

\[ \liminf \sum f_n \leq \sum f. \quad (2) \]

By Th 14.3, \(\liminf \sum f_n \geq \lim \sum f_n\). Thus, (1), (2) and Th 14.4 imply

that \(\lim \sum f_n = \lim \sum f_n = \lim \sum f = \sum f\).

2) Done in class.

3) By definition of \(F\):

\[ f = \lim_{k \to \infty} \sum_{n=1}^{k} S_n. \]

Denote \(f_k = \sum_{n=1}^{k} S_n\). Since \(S_n\) are nonnegative, \(f_{k+1} \geq f_k \geq 0\)

for \(k = 1, 2, \ldots\) and \(f_k \to f\). By the MCT and Th 21.1 (6), we obtain

\[ \sum f = \lim_{k \to \infty} \sum f_k = \lim_{k \to \infty} \sum_{n=1}^{k} S_n = \lim_{k \to \infty} \sum_{n=1}^{k} \sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} S_n. \]

4) Let \(S_n = f \cdot 1_{E_n}\) for \(n = 1, 2, \ldots\). Since \(\{E_n\}\) are disjoint on \(\bigcup_{n=1}^{\infty} E_n = E\) we have:

\[ f = \sum_{n=1}^{\infty} S_n \quad \text{on } E. \]

Since \(f\) is nonnegative, so are \(S_n\), \(n = 1, 2, \ldots\). From the previous problem

\[ \sum f = \sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} \sum f \cdot 1_{E_n} = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} S_n. \]
5) Since \( f_n \geq f_\infty \geq 0 \) on \( E \), we have \( 0 \leq f_n \leq f_1 \) on \( E \) for \( n=1,2,\ldots \). Thus, \( |f_n - f_1| \leq f_1 \) on \( E \) for \( n=1,2,\ldots \), and \( f_1 \) is integrable on \( E \). We can apply the LDCT to the sequence \( \langle f_n \rangle \) and obtain \( \int_E f_n \to \int_E f \).

6) As \( f_n \to f \) a.e. in \([0,1] \), \( |f_n - f| \to 0 \) a.e. in \([0,1] \).

Also for \( n=1,2,\ldots \):
\[
|f_n - f| \leq |f_n| + |f| \leq g + 1 \cdot f \quad \text{a.e. in } [0,1].
\]

But \( f \) is integrable in \([0,1] \) and also is \( 1 \cdot f \).

Thus, \( g + 1 \cdot f \) is integrable in \([0,1] \) as a sum of two integrable functions. We can apply the LDCT (more precisely, Prop. 23.3) to the sequence \( |f_n - f| \) and deduce
\[
\int_{[0,1]} |f_n - f| \to \int_{[0,1]} 0 = 0.
\]