MTH 535 F06 Homework 8 – Selected Solutions

1) Let φ be a simple function. Then $\varphi \cdot \chi_E$ is a simple function and by Def 18.2:

$$\int_E \varphi = \int_{\mathbb{R}} \varphi \cdot \chi_E .$$

Since $\varphi \cdot \chi_E$ vanishes outside of set E of measure 0, Def 18.2 implies easily that $\int_{\mathbb{R}} \varphi \cdot \chi_E = 0$. Thus, $\int_E \varphi = 0$ for any simple function φ.

Def 19.2 gives then $\int_E f = 0$.

2) Let $A = \{x \in E : f(x) > 0\}$. Since $f \geq 0$ a.e. in E, to prove that $f = 0$ a.e. in E we have to show $m(A) = 0$.

Suppose $m(A) > 0$. To obtain a contradiction with the assumption $\int_E f = 0$, we will show that there exists a constant c and a set $B \subseteq A$ such that

$$c > 0 , \quad f \geq c \text{ on } B , \quad m(B) > 0. \qquad (1)$$

Let $B_n = \{x \in A : f(x) \geq \frac{1}{n}\}$ for $n = 1, 2, \ldots$ We have $\bigcup_{n=1}^{\infty} B_n = A$.

Observe that at least one of the sets B_n, say B_{n_0}, is of positive measure. Indeed, if all sets B_n have measure 0, then $m(A) = 0$. Thus, (1) holds with $B = B_{n_0}$, $c = \frac{1}{n_0}$. Thus, by Th 20.1:

$$\int_E f = \int_{E \sim B} f + \int_B f \geq \int_B f \geq \int_B c = c\, m(B) > 0.$$

Contradiction. Thus, $m(A) = 0$ and $f = 0$ a.e. in E.

3) Let M be a positive constant such that

$$|f(x)| \leq M \quad \text{for } x \in E.$$

Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{M}$. For any set $A \subseteq E$, $m(A) < \delta$, we have

$$\left|\int_A f\right| \leq \int_A |f| \leq M \cdot m(A) < \varepsilon.$$

(To obtain the inequalities, we used Th. 20.1).

4) Define $f_n = f \cdot \chi_{E_n}$ for $n = 1, 2, \ldots$ Let $x \in E$. As $\bigcup_{n=1}^{\infty} E_n = E$, $x \in E_{n_0}$ for some $n_0 \in \mathbb{N}$. As $E_n \subseteq E_{n+1}$, $x \in E_n$ for $n \geq n_0$. Thus, $f_n(x) = f(x)$ for $n \geq n_0$ and $f_n \to f$ on E. Let M be a constant such that $|f| \leq M$ on E. Then $|f_n| \leq M$ on E. By the BCT we obtain

$$\int_E f = \lim_{n \to +\infty} \int_E f_n.$$

But $\int_E f_n = \int_E f \cdot \chi_{E_n} = \int_{E_n} f + \int_{E \sim E_n} 0 = \int_{E_n} f$ by Th. 20.1.