MTH 535 Fall 06 Homework 7 – Selected Solutions

#1. Proved in class (Prop 15.3).

#2. Let $\alpha \in \mathbb{R}$ be given. Observe that

$$\{x \in E : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E : f_n(x) \geq \alpha\}.$$

$\{x \in E : f_n(x) \geq \alpha\} \in \mathcal{M}$ by Prop 15.1 as f_n, $n=1,2,\ldots$, are measurable.

Thus $\{x \in E : f(x) \geq \alpha\} \in \mathcal{M}$ as a countable intersection of measurable sets is measurable. Hence, f is measurable by Prop. 15.1.

#3: We use Def 14.1 and Th 14.3. The subsequence

$$a_{n_k} = 2^{(-1)^{2k} \cdot 2k} = 2^{2k}, \quad k=1,2,\ldots$$

converges to $+\infty$. Thus, by Th 14.3, $\overline{\lim}\, a_n = +\infty$. As $a_n \geq 0$ for $n=1,2,\ldots$, $\underline{\lim}\, a_n \geq 0$ by Def 14.1. The subsequence

$$a_{n_m} = 2^{(-1)^{2m+1}(2m+1)} = \frac{1}{2^{2m+1}}, \quad m=1,2,\ldots$$

converges to 0. Hence, $\underline{\lim}\, a_n = 0$ by Th 14.3.

#4: By Th 14.3, there exists a subsequence

$$x_{n_k} + y_{n_k}, \quad k=1,2,\ldots$$

of the sequence $\langle x_n + y_n \rangle_{n=1}^{\infty}$ such that

$$\lim_{k \to +\infty} (x_{n_k} + y_{n_k}) = \overline{\lim}\, (x_n + y_n). \quad (1)$$

As $\langle x_{n_k} \rangle_{k=1}^{\infty}$ is bounded, by Bolzano-Weierstrass Theorem it contains a convergent subsequence:

$$\langle x_{n_{k_m}} \rangle_{m=1}^{\infty}.$$

By (1), the subsequence $\langle x_{n_{k_m}} + y_{n_{k_m}} \rangle_{m=1}^{\infty}$ of $\langle x_{n_k} + y_{n_k} \rangle_{k=1}^{\infty}$ converges to the same limit $(\overline{\lim}\, (x_n + y_n))$. As $\langle x_{n_{k_m}} \rangle_{m=1}^{\infty}$ and $\langle x_{n_{k_m}} + y_{n_{k_m}} \rangle_{m=1}^{\infty}$ both converge, $\langle y_{n_{k_m}} \rangle_{m=1}^{\infty} = \langle x_{n_{k_m}} + y_{n_{k_m}} - x_{n_{k_m}} \rangle_{m=1}^{\infty}$ converges as $m \to +\infty$ and:

$$\lim_{m\to+\infty}(x_{n_{k_m}}+y_{n_{k_m}}) = \lim_{m\to+\infty} x_{n_{k_m}} + \lim_{m\to+\infty} y_{n_{k_m}} =$$

$$= \overline{\lim}\,(x_n+y_n).$$

By Th 14.3,

$$\lim_{m\to+\infty} x_{n_{k_m}} \le \overline{\lim}\, x_n \quad \text{and} \quad \lim_{m\to+\infty} y_{n_{k_m}} \le \overline{\lim}\, y_n .$$

Thus

$$\lim_{m\to+\infty} x_{n_{k_m}} + \lim_{m\to+\infty} y_{n_{k_m}} = \overline{\lim}\,(x_n+y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n.$$

#5. Let $A=\{x\in D : f(x)\geq 0\}$. As f is measurable, $A\in\mathcal{M}$. By #1, χ_A is measurable. Observe that $f^+ = f\cdot\chi_A$. Thus, f^+ is measurable by Prop 15.4.

#6. Let $\varepsilon>0$ be given. From Egroff's Theorem, there exists a set $A_\varepsilon \subseteq E$ such that $m(A_\varepsilon)<\varepsilon$ and $f_n \rightrightarrows f$ on $E\sim A_\varepsilon$. Then, there exists $N\in\mathbb{N}$ such that

$$|f_n(x)-f(x)|<\delta \text{ for all } x\in E\sim A_\varepsilon \text{ and } n\geq N.$$

Thus, for all $n\geq N$, $\{x\in E : |f_n(x)-f(x)|\geq\delta\}\subseteq A_\varepsilon$.

We conclude that

$$m(\{x\in E : |f_n(x)-f(x)|\geq\delta\})<\varepsilon \quad \text{for } n\geq N$$

which gives $m(\{x\in E : |f_n(x)-f(x)|\geq\delta\})\to 0$ as $n\to+\infty$.