MTH 535 Fall 06 Homework 6 - Selected Solutions

#2. Define:

$$F_1 = A_1, \quad F_n = A_n \sim A_{n-1} \quad \text{for } n = 2, 3, \ldots$$

As in class, we prove that the family $\{F_n\}_{n=1}^{\infty}$ is disjoint and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} A_n$. From coutable additivity of the measure, we obtain:

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} m(F_n) = \lim_{n\to+\infty} \sum_{k=1}^{n} m(F_k). \qquad (1)$$

But $$\sum_{k=1}^{n} m(F_k) = m\left(\bigcup_{k=1}^{n} F_k\right) = m\left(\bigcup_{k=1}^{n} A_k\right) = m(A_n). \qquad (2)$$

(The first equality follows from additivity of m, the second from the definition of F_k's, the last from the fact that $A_k \subseteq A_{k+1}$, $k = 1, 2, \ldots$)

(1) and (2) imply

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to+\infty} m(A_n).$$

#1. Proof straightforward from #2. Indeed, let $B_n = \bigcup_{i=1}^{n} A_i$, $n = 1, 2, \ldots$ Then $B_{n+1} \supseteq B_n$, $B_n \in \mathcal{M}$, $n = 1, 2, \ldots$, and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. Thus,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to+\infty} m\left(\bigcup_{i=1}^{n} A_i\right).$$

#3.

Note that

$$E_1 \cup E_2 = (E_1 \sim E_2) \cup (E_2 \sim E_1) \cup (E_2 \cap E_1), \quad E_1 = (E_1 \sim E_2) \cup (E_1 \cap E_2), \quad E_2 = (E_2 \sim E_1) \cup (E_2 \cap E_1),$$

and the collections on the righthand side of each equality are disjoint. From additivity of m we obtain:

$$m(E_1 \cup E_2) = m(E_1 \sim E_2) + m(E_2 \sim E_1) + m(E_2 \cap E_1).$$

Thus:

$$m(E_1 \cup E_2) + m(E_2 \cap E_1) = \big(m(E_1 \sim E_2) + m(E_2 \cap E_1)\big) + \big(m(E_2 \sim E_1) + m(E_2 \cap E_1)\big) = m(E_1) + m(E_2)$$

#4: Done in class.

#5. As $A \cap \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A \cap E_i)$ from subadditivity of m^* we obtain

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i). \qquad (3)$$

From Lemma 11.2 we have for every $k = 1, 2, \ldots$:

$$m^*(A \cap \bigcup_{i=1}^{k} E_i) = \sum_{i=1}^{k} m^*(A \cap E_i).$$

Since $A \cap \bigcup_{i=1}^{\infty} E_i \supseteq A \cap \bigcup_{i=1}^{k} E_i$, we have;

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{k} m^*(A \cap E_i).$$

Since the latter inequality holds for every $k = 1, 2, \ldots$, we have

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i). \qquad (4)$$

Thus, $m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$ by (3) and (4).