MTH 535 Fall 06 Homework 4 – Selected Solutions

3) Prove Prop 16, page 45.

To prove that

$$\bigcap_{F \in \mathcal{C}} F \neq \emptyset \qquad (1)$$

suppose the contrary. That is, suppose $\bigcap_{F \in \mathcal{C}} F = \emptyset$.

Then $\sim \bigcap_{F \in \mathcal{C}} F = \mathbb{R} = \bigcup_{F \in \mathcal{C}} \sim F$. Since all sets F are closed, all sets $\sim F$ are open. Hence, $\{\sim F\}_{F \in \mathcal{C}}$ is an open cover for $\mathbb{R}$. Let $F_0 \in \mathcal{C}$ be bounded. Since $F_0 \subseteq \mathbb{R}$, $\{\sim F\}_{F \in \mathcal{C}}$ is an open cover for F_0. F_0 is closed and bounded. From the Heine-Borel Theorem, there exists a finite subcover

$\{\sim F_1, \sim F_2, \ldots, \sim F_k\}$ for F_0:

$$F_0 \subseteq \bigcup_{i=1}^{k} \sim F_i . \qquad (2)$$

(2) implies easily that $F_0 \cap F_1 \cap \ldots \cap F_k = \emptyset$. Indeed, if $x \in F_0 \cap F_1 \cap \ldots \cap F_k$, then $x \in F_0$ and $x \notin \sim F_i$ for $i = 1, 2, \ldots, k$. Contradiction with (2). Thus, $F_0 \cap F_1 \cap \ldots \cap F_k = \emptyset$ which contradicts the assumption that every finite subcollection of $\mathcal{C}$ has a nonempty intersection.

Thus, (1).

1) Yes. $A \sim B \in \mathcal{A}$; that is $\mathcal{A}$ is closed with respect to the difference of sets. Note first that $\mathcal{A}$ is closed with respect to the intersection. Let $C, D \in \mathcal{A}$. Then $\sim C, \sim D \in \mathcal{A}$ and $\sim C \cup \sim D = \sim(C \cap D) \in \mathcal{A}$. The latter gives $\sim(\sim(C \cap D)) = C \cap D \in \mathcal{A}$. Hence, $\mathcal{A}$ is closed with respect to the intersection. But $A \sim B = A \cap \sim B$. Thus, $A \sim B \in \mathcal{A}$.

5) (c) of Th 8.1 : Let

$$\mathcal{C}_1 = \{(a,b) : a<b, a,b \in \mathbb{R}\}, \mathcal{C}_2 = \{[a,b] : a<b, a,b\in\mathbb{R}\}.$$

We proved in class that $\mathcal{B} = \sigma(\mathcal{C}_1)$. To prove that

$$\mathcal{B} = \sigma(\mathcal{C}_2), \qquad (3)$$

we use Prop 8.1. We have to show that

$$\mathcal{C}_1 \subseteq \sigma(\mathcal{C}_2) \quad \text{and} \quad \mathcal{C}_2 \subseteq \sigma(\mathcal{C}_1). \qquad (4)$$

Take any $(a,b) \in \mathcal{C}_1$. Then

$$(a,b) = \bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, b-\tfrac{1}{n}].$$

As $[a+\frac{1}{n}, b-\frac{1}{n}] \in \mathcal{C}_2$, $(a,b) \in \sigma(\mathcal{C}_2)$ and the first part of (4) is proved. To prove the second part, take any $[a,b] \in \mathcal{C}_2$. Then

$$[a,b] = \bigcap_{n=1}^{\infty} (a-\tfrac{1}{n}, b+\tfrac{1}{n}).$$

Since $(a-\frac{1}{n}, b+\frac{1}{n}) \in \mathcal{C}_1$ and $\sigma(\mathcal{C}_1)$ as any σ-algebra is closed with respect to countable intersections, $[a,b] \in \sigma(\mathcal{C}_1)$. Hence, (4). By Prop 8.1, $\sigma(\mathcal{C}_1) = \mathcal{B} = \sigma(\mathcal{C}_2)$.

6) (d) of Th 8.1. Let

$$\mathcal{C}_3 = \{(a,+\infty) : a \in \mathbb{R}\}.$$

It suffices to show that

$$\mathcal{C}_3 \subseteq \sigma(\mathcal{C}_1) \quad \text{and} \quad \mathcal{C}_1 \subseteq \sigma(\mathcal{C}_3). \qquad (5)$$

Indeed, (5) via Prop 8.1 gives $\sigma(\mathcal{C}_3) = \sigma(\mathcal{C}_1)$. Since $\sigma(\mathcal{C}_1) = \mathcal{B}$, $\sigma(\mathcal{C}_3) = \mathcal{B}$.

To prove (5), observe that for any $a \in \mathbb{R}$:

$$(a, +\infty) = \bigcup_{n=1}^{\infty} (a, n).$$

Thus, $\mathcal{C}_3 \subseteq \sigma(\mathcal{C}_1)$. To prove the second part of (5), take $(a,b) \in \mathcal{C}_1$. We have $(a,+\infty) \in \sigma(\mathcal{C}_3)$ and $(b-\frac{1}{n}, +\infty) \in \sigma(\mathcal{C}_3)$ for all $n=1,2,\dots$ Thus, $\sim(b-\frac{1}{n}, +\infty) = (-\infty, b-\frac{1}{n}] \in \sigma(\mathcal{C}_3)$ for $n=1,2,\dots$. Then $(a,+\infty) \cap (-\infty, b-\frac{1}{n}] \in \sigma(\mathcal{C}_3)$ for $n=1,2,\dots$. Thus, $(a, b-\frac{1}{n}] \in \sigma(\mathcal{C}_3)$ for $n=1,2,\dots$, which gives $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}] \in \sigma(\mathcal{C}_3)$. Thus, (5).