MTH 535 Fall 06 Homework 3 – Seleted Solutions

2) Prop 6.1 and Prop 6.2 follow easily from Prop 5.1 and Prop 5.2 via De Morgan's laws.

4) Let $A = [0,1)$, $B = (1,2]$. Then $A \cap B = \emptyset$. From the definition of closure (Def 7.1), we prove that
$$\bar{A} = [0,1], \quad \bar{B} = [1,2].$$
(It is straightforward, but you should prove it.) Thus $\bar{A} \cap \bar{B} = \{1\} \neq \emptyset$.

5) To prove the $E'$ is closed, we use Th. 6.1. Let $a_n \in E'$ for $n = 1, 2, \ldots$; $a_n \to a_0$ for some $a_0 \in \mathbb{R}$. It suffices to prove that $a_0 \in E'$. If $a_{n_0} = a_0$ for some $n_0 \in \mathbb{N}$, then $a_0 \in E'$ as $a_{n_0} \in E'$. Assume then:
$$a_n \neq a_0 \quad \text{for } n = 1, 2, \ldots$$
To prove that $a_0 \in E'$, we have to prove that $a_0$ is a point of closure of $E \sim \{a_0\}$. In other words, we have to prove that for any $\delta > 0$ we have:
$$(a_0 - \delta, a_0 + \delta) \cap (E \sim \{a_0\}) \neq \emptyset. \quad (1)$$
Let $\delta > 0$ be fixed. As $a_n \to a_0$, there exists $N \in \mathbb{N}$ such that
$$a_N \in (a_0 - \delta, a_0 + \delta).$$
Let $\varepsilon > 0$ be such that $(a_N - \varepsilon, a_N + \varepsilon) \subseteq (a_0 - \delta, a_0 + \delta) \sim \{a_0\}$. Such $\varepsilon$ exists as $a_N \neq a_0$. Since $a_N \in E'$, there exists $p \in E \sim \{a_N\}$ such that $p \in (a_N - \varepsilon, a_N + \varepsilon)$. As $p \neq a_0$, (1) is proved. Hence, $a_0 \in E'$ which gives that $E'$ is closed by Th 6.1.