1) Define a mapping $f: P(N) \to E$ as follows. Let $S \subseteq N$. Then
\[ f(S) = \langle a_n \rangle_{n=1}^\infty, \]
where:
\[ a_n = \left\{ \begin{array}{ll} 1 & , n \in S \\ 0 & , n \notin S. \end{array} \right. \]

$f$ is a bijection. Indeed, let $S \subseteq N$, $T \subseteq N$, $S \neq T$. Denote $f(S) = \langle a_n \rangle_{n=1}^\infty$, $f(T) = \langle b_n \rangle_{n=1}^\infty$. Since $S \neq T$, $S \cap T \neq \emptyset$ or $T \cap S \neq \emptyset$. Suppose $S \cap T \neq \emptyset$ and $n_0 \in S$, $n_0 \notin T$. Then $a_{n_0} = 1$ while $b_{n_0} = 0$. Thus, $\langle a_n \rangle_{n=1}^\infty \neq \langle b_n \rangle_{n=1}^\infty$ and $f$ is an injection. To show that $f$ is a surjection, take $\langle c_n \rangle_{n=1}^\infty \in E$. Let $S \subseteq N$ be such that for every $n \in N$, $n \in S$ if $c_n = 1$ and $n \notin S$ if $c_n = 0$. Then $f(S) = \langle c_n \rangle_{n=1}^\infty$ so $f$ is onto.

2) "$\Rightarrow$" Assume that $\mathcal{A}$ is a $\mathcal{Z}$-algebra, that is, (a), (b), (c) of Prop 4.2 hold. (a) gives (i) and (b) gives (iii). To prove (iii), take $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{A}$. By (b), $\bigcap_{n=1}^\infty A_n \in \mathcal{A}$. By (c), $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$. Using De Morgan's laws, we obtain
\[ \bigcup_{n=1}^\infty A_n = \bigcap_{n=1}^\infty A_n \in \mathcal{A}. \]
Thus, by (b), $\bigcap_{n=1}^\infty A_n \in \mathcal{A}$ and (iii) is proved.

"$\Leftarrow$" Assume that $\mathcal{A}$ satisfies (i) - (iii). We show that (a), (b), (c) of Prop 4.2 hold; that is, $\mathcal{A}$ is a $\mathcal{Z}$-algebra. (i) gives (a), (ii) gives (b). We prove (c) using De Morgan's laws similarly as above.

3) The algebra generated by $\mathcal{B}$ is the finite-cofinite algebra $A_F$. Indeed, any algebra containing $\mathcal{B}$ must contain all finite sets or they are finite unions of singletons. $A$ must also contain all complement of finite sets. Hence, $A$ must contain $A_F$. Thus $A_F$ is the smallest algebra containing $\mathcal{B}$. 
4) We use the following properties of preimages. Let 
\( f : X \to Y \) be given. Then:
(i) \( f^{-1}(\sim A) = \sim f^{-1}(A) \) for any \( A \in \mathcal{Y} \).
(ii) \( f^{-1}(\emptyset) = \emptyset \)
(iii) \( f^{-1}(\bigcup A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A) \) for any \( \mathcal{A} \subseteq \mathcal{P}(Y) \).

Some of these properties were proved in Homework 1, #6. The others are easy to prove from the definition of the preimage.

We prove that \( \mathcal{A} \) is an algebra from Prop 4.2. \( \emptyset \in \mathcal{D} \) as \( \mathcal{D} \) is a \( \delta \)-algebra. Hence, by (ii) \( \emptyset \in \mathcal{A} \). Take \( B \in \mathcal{A} \).

Then \( B = f^{-1}(A) \) for some \( A \in \mathcal{D} \). As \( \mathcal{D} \) is a \( \delta \)-algebra, \( \sim A \in \mathcal{D} \). By (i), \( f^{-1}(\sim A) = \sim f^{-1}(A) = \sim B \), thus \( \sim B \in \mathcal{A} \).

Let \( B_1, B_2, \ldots, B_n, \ldots \in \mathcal{A} \). Then \( B_i \in f^{-1}(A_i) \) for some \( A_i \in \mathcal{D} \), for \( i = 1, 2, \ldots \). As \( \mathcal{D} \) is a \( \delta \)-algebra, \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{D} \).

By (iii), \( f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) = \bigcup_{i=1}^{\infty} B_i \), thus \( \bigcup_{i=1}^{\infty} B_i \in \mathcal{A} \).

5) Let \( X, Y \) be countable. Let 
\( X = \{ x_1, x_2, \ldots, x_n, \ldots \} \) \( , \ Y = \{ y_1, y_2, \ldots, y_m, \ldots \} \).

Then 
\( X \times Y = \{ < x_n, y_m > : n, m \in \mathbb{N} \} \).

Denote for each \( n = 1, 2, \ldots \)
\( A_n = \{ < x_n, y_m > : m = 1, 2, \ldots \} \).

Then each \( A_n \) is countable and \( X \times Y = \bigcup_{n=1}^{\infty} A_n \). As a countable union of countable sets, \( X \times Y \) is countable.
6) Define the mapping \( f : \mathcal{C} \rightarrow \mathbb{Q} \times \mathbb{Q} \) as follows:

\[
f((q_1, q_2)) = <q_1, q_2>.
\]

Clearly, \( f \) is an injection. Hence, \( \mathcal{C} \) is equinumerous with a subset of \( \mathbb{Q} \times \mathbb{Q} \). \( \mathbb{Q} \times \mathbb{Q} \) is countable by the previous problem. Thus, every subset of \( \mathbb{Q} \times \mathbb{Q} \) is countable.