Homework 2 – Solutions MTH 535 Fall 2006

1) Define a mapping $f: \mathcal{P}(\mathbb{N}) \to E$ as follows. Let $S \subseteq \mathbb{N}$. Then $f(S) = \langle a_n \rangle_{n=1}^{\infty}$ where:

$$a_n = \begin{cases} 1, & n \in S \\ 0, & n \notin S. \end{cases}$$

f is a bijection. Indeed, let $S \subseteq \mathbb{N}$, $T \subseteq \mathbb{N}$, $S \neq T$. Denote $f(S) = \langle a_n \rangle_{n=1}^{\infty}$, $f(T) = \langle b_n \rangle_{n=1}^{\infty}$. Since $S \neq T$, $S \sim T \neq \emptyset$ or $T \sim S \neq \emptyset$. Suppose $S \sim T \neq \emptyset$ and $n_0 \in S, n_0 \notin T$. Then $a_{n_0} = 1$ while $b_{n_0} = 0$. Thus, $\langle a_n \rangle_{n=1}^{\infty} \neq \langle b_n \rangle_{n=1}^{\infty}$ and f is an injection. To show that f is a surjection, take $\langle c_n \rangle_{n=1}^{\infty} \in E$. Let $S \subseteq \mathbb{N}$ be such that for every $n \in \mathbb{N}$, $n \in S$ if $c_n = 1$ and $n \notin S$ if $c_n = 0$. Then $f(S) = \langle c_n \rangle_{n=1}^{\infty}$ so f is onto.

2) "$\Rightarrow$" Assume that $\mathcal{A}$ is a σ-algebra; that is, (a), (b), (c) of Prop 4.2 hold. (a) gives (i) and (b) gives (ii). To prove (iii), take $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{A}$. By (b), $\sim A_1, \sim A_2, \ldots, \sim A_n, \ldots \in \mathcal{A}$. By (c), $\bigcup_{n=1}^{\infty} \sim A_n \in \mathcal{A}$. Using DeMorgan's Laws, we obtain $\bigcup_{n=1}^{\infty} \sim A_n = \sim \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$. Thus, by (b) $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$ and (iii) is proved.

"$\Leftarrow$" Assume that $\mathcal{A}$ satisfies (i)–(iii). We show that (a), (b), (c) of Prop 4.2 hold; that is, $\mathcal{A}$ is a σ-algebra. (i) gives (a), (ii) gives (b). We prove (c) using DeMorgan's Laws similarly as above.

3) The algebra generated by $\mathcal{S}$ is the finite-cofinite algebra $\mathcal{A}_F$. Indeed, any algebra $\mathcal{A}$ containing $\mathcal{S}$ must contain all finite sets as they are finite unions of singletons. $\mathcal{A}$ must also contain all complement of finite sets. Hence, $\mathcal{A}$ must contain $\mathcal{A}_F$. Thus $\mathcal{A}_F$ is the smallest algebra containing $\mathcal{S}$.

4) We use the following properties of preimages. Let $f: X \to Y$ be given. Then:

(i) $f^{-1}(\sim A) = \sim f^{-1}(A)$ for any $A \in Y$.

(ii) $f^{-1}(\emptyset) = \emptyset$

(iii) $f^{-1}\left(\bigcup_{A \in \mathcal{C}} A\right) = \bigcup_{A \in \mathcal{C}} f^{-1}(A)$ for any $\mathcal{C} \subseteq \mathcal{P}(Y)$.

Some of these properties were proved in Homework 1, # 6. The others are easy to prove from the definition of the preimage.

We prove that $\mathcal{A}$ is an algebra from Prop 4.2. $\emptyset \in \mathcal{D}$ as $\mathcal{D}$ is a σ-algebra. Hence, by (ii) $\emptyset \in \mathcal{A}$. Take $B \in \mathcal{A}$. Then $B = f^{-1}(A)$ for some $A \in \mathcal{D}$. As $\mathcal{D}$ is a σ-algebra, $\sim A \in \mathcal{D}$. By (i), $f^{-1}(\sim A) = \sim f^{-1}(A) = \sim B$. Thus $\sim B \in \mathcal{A}$.

Let $B_1, B_2, \dots, B_n, \dots \in \mathcal{A}$. Then $B_i \in f^{-1}(A_i)$ for some $A_i \in \mathcal{D}$, for $i = 1, 2, \dots$. As $\mathcal{D}$ is σ-algebra, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{D}$. By (iii), $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) = \bigcup_{i=1}^{\infty} B_i$. Thus, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$.

5) Let X, Y be countable. Let

$$X = \{x_1, x_2, \dots, x_n, \dots\}, \quad Y = \{y_1, y_2, \dots, y_m, \dots\}.$$

Then

$$X \times Y = \{\langle x_n, y_m \rangle : n, m \in \mathbb{N}\}.$$

Denote for each $n = 1, 2, \dots$

$$A_n = \{\langle x_n, y_m \rangle : m = 1, 2, \dots\}.$$

Then each A_n is countable and $X \times Y = \bigcup_{n=1}^{\infty} A_n$. As a countable union of countable sets, $X \times Y$ is countable.

6) Define the mapping $f : \mathcal{C} \rightarrow \mathbb{Q} \times \mathbb{Q}$ as follows:

$$f((q_1, q_2)) = \langle q_1, q_2 \rangle.$$

Clearly, f is an injection. Hence, $\mathcal{C}$ is equinumerous with a subset of $\mathbb{Q} \times \mathbb{Q}$. $\mathbb{Q} \times \mathbb{Q}$ is countable by the previous problem. Thus, every subset of $\mathbb{Q} \times \mathbb{Q}$ is countable.