Homework 1 – Solutions

1) From the assumptions we have:

$$a_n \le a_{n+1}, \qquad b_{n+1} \le b_n \qquad \text{for all } n=1,2,\dots \quad . (1)$$

Also

$$\lim_{n\to+\infty} (b_n - a_n) = 0 \quad . \quad (2)$$

As $[a_n, b_n] \subseteq [a_1, b_1]$ for all $n=1,2,\dots$, both sequences $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ are bounded and, by (1), monotone.

Hence, they are convergent and:

$$\lim_{n\to+\infty} a_n = a \quad , \quad a = \sup\{a_n : n \in \mathbb{N}\}, \qquad (3)$$

$$\lim_{n\to+\infty} b_n = b \quad , \quad b = \inf\{b_n : n \in \mathbb{N}\}. \qquad (4)$$

By (2), (3), and (4):

$$\lim_{n\to+\infty} (b_n - a_n) = \lim_{n\to+\infty} b_n - \lim_{n\to+\infty} a_n = a - b = 0.$$

Thus $a = b$. Let $c = a = b$. We shall prove that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{c\} . \qquad (5)$$

By (3) and (4) we have:

$$a_n \le c \le b_n \quad \text{for } n = 1,2,\dots .$$

Thus, $\{c\} \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n]$. Suppose that for some $d \in \mathbb{R}$, $d \ne c$, $d \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. Let $|c-d| = \varepsilon > 0$. From (2), for some $N \in \mathbb{N}$, $b_n - a_n < \varepsilon$ whenever $n \ge N$. But $c, d \in [a_n, b_n]$ for all $n \in \mathbb{N}$ which implies $b_n - a_n \ge \varepsilon$ for all $n \in \mathbb{N}$. Contradiction. Hence, (5).

2) Suppose E is countable. Let $f : \mathbb{N} \to E$ be a bijection. Denote $s_n = f(n)$ for $n=1,2,\dots$. Then:

$$E = \{s_1, s_2, \dots, s_n, \dots\}$$

where each s_n is a sequence of 0's and 1's:

$$s_1 = \langle a_1^1, a_2^1, a_3^1, \ldots, a_m^1, \ldots \rangle$$

$$s_2 = \langle a_1^2, \quad \ldots \quad, a_m^2, \ldots \rangle$$

$$\vdots$$

$$s_n = \langle a_1^n, a_2^n, \ldots, a_m^n, \ldots \rangle .$$

$$\vdots$$

Define a sequence $b = \langle q_i \rangle_{i=1}^{\infty}$ as follows:

$$q_i = \begin{cases} 0 & \text{if } a_i^i = 1 \\ 1 & \text{if } a_i^i = 0 . \end{cases}$$

Then $b \neq s_n$ for every $n \in \mathbb{N}$ and $b \in E$. Contradiction. Thus E is not countable.

3) Let a set X and a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ be given.
We shall prove that

$$\sim \bigcup_{A \in \mathcal{C}} A = \bigcap_{A \in \mathcal{C}} \sim A . \qquad (6)$$

(The other part is similar.)

Let $x \in \sim \bigcup_{A \in \mathcal{C}} A$. Then $x \notin \bigcup_{A \in \mathcal{C}} A$. Hence, $x \notin A$ for every $A \in \mathcal{C}$, which gives $x \in \sim A$ for every $A \in \mathcal{C}$. Thus, $x \in \bigcap_{A \in \mathcal{C}} \sim A$ and the inclusion $\sim \bigcup_{A \in \mathcal{C}} A \subseteq \bigcap_{A \in \mathcal{C}} \sim A$ is proved.

Let $x \in \bigcap_{A \in \mathcal{C}} \sim A$. Then $x \in \sim A$ for every $A \in \mathcal{C}$ which implies $x \notin A$ for every $A \in \mathcal{C}$. The latter gives $x \notin \bigcup_{A \in \mathcal{C}} A$ and hence $x \in \sim \bigcup_{A \in \mathcal{C}} A$. Therefore, $\bigcap_{A \in \mathcal{C}} \sim A \subseteq \sim \bigcup_{A \in \mathcal{C}} A$ and (6) is proved.

4) Skipped

5) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Take $A = (-1, 0)$, $B = (0, 1)$. Then $f(A \cap B) = f(\emptyset) = \emptyset$.
Yet, $f(A) = f(B) = \{0\}$. Thus $f(A) \cap f(B) = \{0\} \neq \emptyset$.

6) We prove (c). (b) and (a) are similar.
Let $B \subseteq Y$. Let $x \in f^{-1}(\sim B)$. Then by the definition of the preimage, $f(x) \in \sim B$. Hence, $f(x) \notin B$ and $x \notin f^{-1}(B)$.
Thus $x \in \sim f^{-1}(B)$ and the inclusion:

$$f^{-1}(\sim B) \subseteq \sim f^{-1}(B)$$

is proved. To prove the opposite inclusion, take $x \in \sim f^{-1}(B)$. Then $x \notin f^{-1}(B)$ which implies $f(x) \notin B$. Hence, $f(x) \in \sim B$ and $x \in f^{-1}(\sim B)$. Therefore, $\sim f^{-1}(B) \subseteq f^{-1}(\sim B)$ and (c) is proved.