(23) Recall: f is integrable over E if

$$\int_E f = \int_E f^+ - \int_E f^-$$

is defined and finite.

<u>Prop 23.1</u>: Let f be a measurable function defined on a measurable set E. Then f is integrable in E if and only if $|f|$ is integrable in E. If they are both integrable, then:

$$\left| \int_E f \right| \leq \int_E |f|. \qquad (1)$$

▲

<u>Proof</u>: Assume that f is integrable on E. Then, by Remark 22.1, f^+ and f^- are integrable on E. Thus by Th 21.1 (ii), $f^+ + f^-$ is integrable on E. But

$$|f| = f^+ + f^-. \qquad (2)$$

Hence, $|f|$ is integrable. Assume that $|f|$ is integrable on E. From (2), $0 \leq f^+ \leq |f|$. Hence, by Th 21.1:

$$0 \leq \int_E f^+ \leq \int_E |f| < +\infty.$$

Thus f^+ is integrable. Similarly we obtain that f^- is integrable and so is f by Remark 22.1.

(1) follows from Th 22. (iii) and (i). Indeed:

$$-|f| \leq f \leq |f|$$

Thus

$$-\int_E |f| \leq \int_E f \leq \int_E |f|.$$

The latter implies (1).

■

Therefore, for the Lebegue integral integrability and absolute integrability are equivalent. It is not so for the Riemann integral.

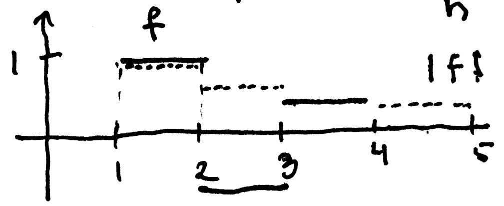
Ex : Define $f : [0,1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in [0,1] \sim \mathbb{Q}. \end{cases}$$

f is not Riemann integrable in $[0,1]$ but $|f| \equiv 1$ is. f and $|f|$ are both Lebesque integrable on $[0,1]$ with the integral 0.

Ex : Consider $f : [1, +\infty) \to \mathbb{R}$ defined as :

$$f(x) = \frac{(-1)^{n+1}}{n}, \quad x \in [n, n+1), \; n = 1, 2, \dots .$$



It is easy to prove that

$$R\int_1^{+\infty} f(x)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} < +\infty .$$

Yet, $L\int_{[1,+\infty)} |f| = \sum_{n=1} \frac{1}{n} = +\infty$. Thus, $|f|$ is not Lebesgue integrable in $[1,+\infty]$ and f is not either. Hence, for functions which are not necesserily nonnegative, summability in the sense of Riemann and summability in the sense of Lebesgue are different.

Prop 23.2 Let f be integrable over E, g be measurable and defined on E. If

$$|g| \le f \text{ a.e. on } E,$$

then g is integrable on E.

Proof : By Th 21.1 :

$$0 \le \int_E |g| \le \int_E f < +\infty .$$

So $|g|$ is integrable and, by Prop 23.1, so is g.

Lemma 23.1 : Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence from $\mathbb{R}$, $c \in \mathbb{R}$. Then

(a) $\underline{\lim}\,(c - a_n) = c - \overline{\lim}\, a_n$ (b) $\underline{\lim}\,(c + a_n) = c + \underline{\lim}\, a_n$.

Proof : Easy from Th 14.3.

<u>Th 23.1 (The Lebesgue Dominated Convergence Theorem):</u> Let g be integrable over E. Let f_n, $n=1,2,\dots$, be measurable functions defined on E such that:

$$|f_n| \le g \quad \text{on } E \text{ for } n=1,2,\dots . \qquad (3)$$

Assume that for some function f defined on E we have:

$$f_n \to f \quad \text{a.e. on } E \qquad (4)$$

Then $f, f_n, n=1,2,\dots,$ are integrable on E and

$$\int_E f = \lim_{n\to+\infty} \int_E f_n . \qquad (5)$$

▲

<u>Proof</u>: f is measurable as an a.e. limit of measurable functions. By Prop. 23.2, f_n are all integrable on E for $n=1,2,\dots$. Observe that $|f_n| \to |f|$ a.e. on E by (4). Thus from (3):

$$|f| \le g \quad \text{a.e. on } E.$$

By Prop 23.2, f is integrable on E. To prove (5) observe that by (3):

$$-g \le f_n \le g \quad \text{on } E \text{ for } n=1,2,\dots .$$

Hence:

$$g - f_n \ge 0 \quad \text{and} \quad f_n + g \ge 0 \text{ on } E \text{ for } n=1,2,\dots . \quad (6)$$

We have also:

$$g - f_n \to g - f \text{ a.e. on } E, \quad f_n + g \to f + g \text{ a.e. on } E. \quad (7)$$

By (6) and (7) we can apply Fatou's Lemma to both sequences. We obtain:

$$\int_E (g-f) \overset{\text{Th 22.1}}{=} \int_E g - \int_E f \overset{\text{F.L.}}{\le} \underline{\lim} \int_E (g - f_n) \overset{\text{Th 22.1}}{=} \underline{\lim} \left(\int_E g - \int_E f_n \right).$$

From Lemma 23.1:

$$\int_E g - \int_E f \leq \int_E g - \overline{\lim} \int_E f_n .$$

Thus:

$$\int_E f \geqslant \overline{\lim} \int_E f_n . \quad (8)$$

Applying Fatou's Lemma to $\langle g + f_n \rangle$, we obtain:

$$\int_E (g+f) = \int_E g + \int_E f \leq \underline{\lim} \int_E (g+f_n) = \underline{\lim} \left(\int_E g + \int_E f_n \right).$$

By Lemma 23.1:

$$\int_E g + \int_E f \leq \int_E g + \underline{\lim} \int_E f_n .$$

Thus

$$\int_E f \leq \underline{\lim} \int_E f_n . \quad (9).$$

(9) and (8) give:

$$\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f .$$

The latter implies (5).

■

<u>Prop 23.3</u>: Th 23.1 remains valid for extended real-valued functions f, f_n, g, $n = 1, 2, \ldots$, with the assumption

$$|f_n| \leq g \text{ on } E \text{ for } n = 1, 2, \ldots$$

replaced by

$$|f_n| \leq g \text{ a.e. on } E.$$

▲