Homework 9 Fall 05 - Solutions

4) f continuous on $[a,b]$. Thus, by the Max-Min Theorem there exists $r, p \in [a,b]$ such that:

$$f(r) \leq f(x) \leq f(p) \text{ for all } x \in [a,b]. \quad (1)$$

Let $R = \{ f(x) : x \in [a,b] \}$ be the range of f. By (1), $R \subseteq [f(r), f(p)]$. Let $v \in (f(r), f(p))$. By the IVT, there exists c in $[r,p]$ or $[p,r]$ such that $f(c) = v$. Thus, $v \in R$ and $[f(r), f(p)] \subseteq R$. Hence, $R = [f(r), f(p)]$.

6) Let $f : [a,b] \to [a,b]$ be continuous. Consider $h(x) = f(x) - x$. It suffices to show that there exists $x_0 \in [a,b]$ such that $h(x_0) = 0$. (Then, $f(x_0) = x_0$ and x_0 is a fixed point of f.)

Observe, that since $a \leq f(x) \leq b$ for all $x \in [a,b]$, we have:

$$h(a) = f(a) - a \geqslant 0 \quad \text{and} \quad h(b) = f(b) - b \leq 0.$$

If the equality holds in one of the above inequalities we can take $x_0 = a$, or, respectively, $x_0 = b$. Suppose that both inequalities are sharp; that is:

$$h(a) > 0 \quad , \quad h(b) < 0.$$

Since h is continuous in $[a,b]$, by the IVT there exists $x_0 \in [a,b]$ such that $h(x_0) = 0$.

1) We have to show that

$$\underset{\varepsilon_0 > 0}{\exists} \ \underset{\delta > 0}{\forall} \ \underset{x_\delta, y_\delta \in (0,+\infty)}{\exists} \left(|x_\delta - y_\delta| < \delta \ \wedge \ |f(x_\delta) - f(y_\delta)| \geqslant \varepsilon_0 \right)$$

Let $\varepsilon_0 = 1$. Let $\delta > 0$ be given. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Let $x_\delta = n$, $y_\delta = (n + \frac{1}{n})$. Then $x_\delta, y_\delta \in (0, +\infty)$,

$$|x_\delta - y_\delta| = \tfrac{1}{n} < \delta \quad \text{and} \quad |f(x_\delta) - f(y_\delta)| = \left(n + \tfrac{1}{n}\right)^2 - n^2 = 2 + \tfrac{1}{n^2} > \varepsilon_0.$$

Thus, $f(x) = x^2$ is not uniformly continuous on $(0, +\infty)$.

2) Th 20.1 does not apply directly as the interval $[1,+\infty)$ is not bounded. To prove that $\frac{1}{x}$ is uniformly continuous on $[1,+\infty)$, we have to show that:

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,y\in[1,+\infty)}\ \left(|x-y|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<\varepsilon\right). \quad (2)$$

Let $\varepsilon>0$ be fixed. Take $K=\frac{1}{\sqrt{\varepsilon}}$. Since $\frac{1}{x}$ is continuous on the interval $[1,K]$, $\frac{1}{x}$ is uniformly continuous on $[1,K]$. Hence, there exists $\delta_1>0$ such that

$$\forall_{x,y\in[1,K]}\ \left(|x-y|<\delta_1 \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<\varepsilon\right). \quad (3)$$

Take $\delta=\min\{\frac{1}{2},\delta_1\}$. Then from (3)

$$\forall_{x,y\in[1,K]}\ \left(|x-y|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<\varepsilon\right). \quad (4)$$

As $\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{|x-y|}{|xy|}$ and $x>K$ and $y>K$ implies $\frac{1}{xy}<\varepsilon$, we have $\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{|x-y|}{xy}<\varepsilon$ whenever $|x-y|<1$ and $x,y\in(K,+\infty)$. The latter and (4) give:

$$\forall_{x,y\in[1,+\infty)}\ \left(|x-y|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<\varepsilon\right).$$

Thus $g(x)=\frac{1}{x}$ is uniformly continuous on $[1,+\infty)$.

5) Is straightforward from Def 21.1 and Th 16.1.

3) Suppose f is not constant on $(-\infty,+\infty)$. Then for some $a,b\in(-\infty,+\infty)$, we have $f(a)\neq f(b)$. Without any loss of generality, we may assume $a<b$. We have $f(a)<f(b)$ or $f(b)>f(a)$. From the density of irrationals there exists v between $f(a)$ and $f(b)$ such that $v\in\mathbb{R}\setminus\mathbb{Q}$. As f is continuous on $[a,b]$, the Intermediate Value Theorem implies that for some $c\in(a,b)$, $f(c)=v$. Contradiction as f takes only rational values.