1) We use Def 18.2. Let $M>0$ be given. Since $\lim_{x\to+\infty} g(x)=0$ applying Def 18.1 with $\varepsilon = \frac{m}{M}$ we obtain that there exists $b>a$ such that

$$|g(x)-0| = g(x) < \frac{m}{M} \quad \text{for all } x>b. \qquad (1)$$

(1) implies that for all $x>b$:

$$\frac{f(x)}{g(x)} \geqslant \frac{m}{g(x)} > M.$$

As $M>0$ was arbitrary, $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty$.

2) (a) Let $f$ be defined in some interval $(-\infty, a)$, $a\in\mathbb{R}$, $L\in\mathbb{R}$. We say that

$$\lim_{x\to-\infty} f(x) = L$$

if

$$\underset{\varepsilon>0}{\forall}\ \underset{b<a}{\exists}\ \underset{x\in(-\infty,a)}{\forall}\ (x<b \Rightarrow |f(x)-L|<\varepsilon)$$

(b) Let $f$ be defined in some open interval $(c,a)$, $a>c$. We say that

$$\lim_{x\to c+} f(x) = -\infty$$

if

$$\underset{M>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in(c,a)}{\forall}\ (c<x<c+\delta \Rightarrow f(x)<-M).$$

(c) Let $f$ defined in some interval $(-\infty,a)$, $a\in\mathbb{R}$. We say

$$\lim_{x\to-\infty} f(x) = -\infty$$

if

$$\underset{M>0}{\forall}\ \underset{b<a}{\exists}\ \underset{x\in(-\infty,a)}{\forall}\ (x<b \Rightarrow f(x)<-M).$$

3) Since $f$ is continuous on $I$ and $c$ is an interior point of $I$ by Th. 17.5, $\lim_{x\to c} f(x) = f(c)$. Using Def. 14.4 with $\varepsilon = \frac{f(c)}{2}$ we obtain that there exists $\delta_1>0$ such that

$$|f(x)-f(c)| < \frac{f(c)}{2} \quad \text{for all } x\in I\cap(c-\delta_1, c+\delta_1).$$

Thus

$$-\frac{f(c)}{2}+f(c) < f(x) < f(c)+\frac{f(c)}{2} \quad \text{for all } x\in I\cap(c-\delta_1,c+\delta_1). \qquad (2)$$

Let $\delta_0>0$ be such that $(c-\delta_0, c+\delta_0)\subseteq I$. Let $\delta = \min\{\delta_0,\delta_1\}$. By (2):

$f(x) > -\frac{f(c)}{2} + f(c) = \frac{f(c)}{2} > 0$ for all $x \in (c-\delta, c+\delta)$.

We can take $N = (c-\delta, c+\delta)$.

4) Let $I$ be an interval, $f$ be defined and continuous in $I$. Let $c$ be an interior point of $I$. Assume that $f(c) < 0$. Then there exists a nbhd $N$ of $c$ such that $N \subseteq I$ and

$$f(x) < 0 \quad \text{for all } x \in N.$$

Proof very similar to the proof of the previous proposition.

5) If $f(c) = 0$ nothing can be said about the sign of $f(x)$ in a nbhd of $c$: in any deleted nbhd of $c$ $f$ can be positive, negative, constantly 0, or changing sign. Examples are easy to give and you should be able to give them.

6) We use Def 18.3. $f(x) = \frac{1}{(x-1)^2}$ is defined in $N_d = (-\infty, +\infty) \setminus \{1\}$.

Let $M > 0$. Take $\delta = \frac{1}{\sqrt{M}}$. Then

$$\frac{1}{(x-1)^2} > M \quad \text{whenever } 0 < |x-1| < \delta.$$

7) By contradiction. Assume that there exists $r \in (0, 1)$ such that $f(r) \neq 0$. Then $f(r) > 0$ or $f(r) < 0$. In either case, Prop H8.1 or Prop H8.2 give that there exists a nbhd $(r-\delta, r+\delta) \subseteq (0, 1)$ such that $f(x) \neq 0$ for all $x \in (r-\delta, r+\delta)$. Contradiction as $(r-\delta, r+\delta)$ contains a rational number.

8) Consider the function $h = f - g$. By Th. 18.3 $h$ is continuous in $\mathbb{R}$. Also, $h(a) = f(a) - g(a) > 0$. By Prop H8.1, there exists a nbhd $N$ of $a$ such that $h(x) = f(x) - g(x) > 0$ for all $x \in N$. Thus $f(x) > g(x)$ for all $x \in N$.