MTH 435 Homework 7 – Solutions F05

1) Denote $N_d = I \setminus \{c\}$. N_d is a deleted nbhd of c, f is defined on N_d.
To prove that $L \le M$ assume otherwise; $L > M$. Take $\varepsilon = L - M$. Then $\varepsilon > 0$. Hence, there exists $\delta > 0$ such that for any $x \in (c-\delta, c+\delta) \cap N_d$ $|f(x) - L| < \varepsilon$. Hence, $-L+M < f(x) - L < L - M$ for any $x \in (c-\delta, c+\delta) \cap N_d$. In particular, $f(x) > M$ for all such x. Contradiction as $f(x) \le M$ for all $x \in N_d$ by assumption. Hence, $L \le M$. We prove $L \ge m$ similarly.

2) Denote again $N_d = I \setminus \{c\}$. To prove that $\lim_{x\to c} f(x) = L$, take $\varepsilon > 0$.
From Def. 14.4, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$L - \varepsilon < h(x) < L + \varepsilon$ for all $x \in (c-\delta_1, c+\delta_1) \cap N_d$,

$L - \varepsilon < g(x) < L + \varepsilon$ for all $x \in (c-\delta_2, c+\delta_2) \cap N_d$.

Take $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and for any $x \in (c-\delta, c+\delta) \cap N_d$ we have

$$L - \varepsilon < g(x) \le f(x) \le h(x) < L + \varepsilon.$$

Hence,

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta, \; x \in N_d.$$

Therefore $\lim_{x\to c} f(x) = L$.

3) $f(x)$ is defined in $N_d = \mathbb{R} \setminus \{0\}$. Also:

$$-x^2 \le x^2 \cos\left(\tfrac{1}{x}\right) \le x^2 \quad \text{for all } x \in N_d.$$

As $\lim_{x\to 0} x = 0$, by Th 16.1, $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}(-x^2) = 0$. By the Squeeze Theorem that we have just proved $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

4) "$\Rightarrow$" Assume that $\lim_{x\to c+} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty}$ be such that $x_n \in (c, a)$ for $n = 1, 2, \ldots$, $x_n \to c$ as $n \to +\infty$. To prove that $f(x_n) \to L$ as $n \to +\infty$ take $\varepsilon > 0$. By the assumption there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta, \; x \in (c, a). \quad (1)$$

Since $x_n \to c$ as $n \to +\infty$ there exists $N \in \mathbb{N}$ such that

$$|x_n - c| < \delta \quad \text{whenever} \quad n \ge N, \quad (2)$$

As $x_n \in (c, a)$, (2) gives

$$c < x_n < c + \delta, \; x_n \in (c, a), \text{ for } n \ge N. \quad (3)$$

(3) and (1) imply:

$$|f(x_n) - L| < \varepsilon \quad \text{whenever } n \geq N.$$

Therefore, $f(x_n) \to L$ as $n \to +\infty$.

"$\Leftarrow$" Assume that for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in (c, a)$ for $n \in \mathbb{N}$ and $x_n \to c$ as $n \to +\infty$ we have $f(x_n) \to L$ as $n \to +\infty$. Suppose that $\lim\limits_{x \to c} f(x)$ is not L. Then:

$$\underset{\varepsilon_0 > 0}{\exists} \ \underset{\delta > 0}{\forall} \ \underset{x \in (c,a)}{\exists} \ (c < x < c + \delta \ \wedge \ |f(x) - L| \geq \varepsilon_0). \quad (4)$$

Take and fix $\varepsilon_0 > 0$ which satisfies (4). (4) implies, in particular, that for every $\delta_n = \frac{1}{n}$, $n = 1, 2, \ldots$, there exists $x_n \in (c, a)$ such that

$$c < x_n < c + \delta_n \quad \text{and} \quad |f(x_n) - L| \geq \varepsilon_0. \quad (5)$$

(5) gives that $x_n \to c$ as $n \to +\infty$ and $\lim\limits_{n \to +\infty} f(x_n)$ is not L. Contradiction. Hence, $\lim\limits_{x \to c} f(x) = L$.

5) <u>Th H7.2</u> Let $c \in \mathbb{R}$, $L \in \mathbb{R}$. Assume that f is defined in an open interval (b, c). Then

$$\lim_{x \to c-} f(x) = L$$

if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in (b, c)$ for $n \in \mathbb{N}$ and $x_n \to c$ as $n \to +\infty$ we have $f(x_n) \to L$ as $n \to +\infty$. ▲