1) Assume first that $\{a_n\}_{n=1}^{\infty}$ is increasing. If $\{a_n\}_{n=1}^{\infty}$ is bounded, then for some $L \in \mathbb{R}$, $\lim_{n\to+\infty} a_n = L$ by Th 11.1 and the proposition is proved. Suppose that $\{a_n\}_{n=1}^{\infty}$ is not bounded. Since $\{a_n\}_{n=1}^{\infty}$ is increasing it is bounded below by a_1. Hence, $\{a_n\}_{n=1}^{\infty}$ must be not bounded above. We shall show that

$$\lim_{n\to+\infty} a_n = +\infty. \qquad (1)$$

Indeed, take $M \in \mathbb{R}$, $M > 0$. Since $\{a_n\}_{n=1}^{\infty}$ is not bounded above, there exists $N \in \mathbb{N}$ such that $a_N > M$. As $\{a_n\}_{n=1}^{\infty}$ is increasing, $a_n \geq a_N > M$ for all $n \geq N$ and (1) is proved.

In the case when $\{a_n\}_{n=1}^{\infty}$ is decreasing the proof is very similar so we skip it.

2) Assume $\{a_n\}_{n=1}^{\infty}$ is not bounded above. We shall define a subsequence $\{a_{m_n}\}_{n=1}^{\infty}$ such that $a_{m_n} \to +\infty$ as $n \to +\infty$ by induction. Since $\{a_n\}_{n=1}^{\infty}$ is not bounded above, there exists $m_1 \in \mathbb{N}$ such that

$$a_{m_1} > 1.$$

Choose $m_2 \in \mathbb{N}$ such that

$$m_2 > m_1 \quad \text{and} \quad a_{m_2} > 2.$$

Such m_2 exists. Indeed, suppose it doesn't. Then $a_n \leq 2$ for all $n > m_1$ and the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above by

$$\max\{|a_1|, \dots, |a_{m_1}|, 2\}$$

which contradicts our assumption. Suppose for a given $k \in \mathbb{N}$ we have defined $m_1, m_2, \dots, m_k$ such that:

$$m_1 < m_2 < \dots < m_k, \qquad a_{m_i} > i \text{ for } i = 1, \dots, k.$$

Choose $m_{k+1} > m_k$ and such $a_{m_{k+1}} > k+1$. Such $m_{k+1} \in \mathbb{N}$ exists as othewise $\{a_n\}_{n=1}^{\infty}$ would be bounded above by

$$\max\{|a_1|, \dots, |a_{m_k}|, k+1\}.$$

By induction we can choose m_n for all $n \in \mathbb{N}$ in such a way that:

$$m_1 < m_2 < \dots < m_n < m_{n+1} < \dots \quad \text{and} \quad a_{m_n} > n \text{ for all } n \in \mathbb{N}.$$

Hence, $\{a_{m_n}\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$. The condition $a_{m_n} > n$ for all $n \in \mathbb{N}$ implies easily that $\lim_{n\to+\infty} a_{m_n} = +\infty$. (Using Def 11.1.)

Prop H6.1: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence which is not bounded below. Then $\{a_n\}_{n=1}^{\infty}$ has a subsequence $\{a_{p_n}\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} a_{p_n} = -\infty$.

3) Let $\{x_n\}_{n=1}^{\infty} = 0, 1, 2, 3, \ldots, n, \ldots$, $\{y_n\}_{n=1}^{\infty} = 0, 0, 0, \ldots$. That is, $x_n = n$, $y_n = 0$ for all $n \in \mathbb{N}$. Then the set of elements $\{y_n : n \in \mathbb{N}\} = \{0\}$ is contained in the set of elements $\{x_n : n \in \mathbb{N}\} = \mathbb{N} \cup \{0\}$. Yet, $\{y_n\}_{n=1}^{\infty}$ is not a subsequence of $\{x_n\}_{n=1}^{\infty}$. Indeed, $y_n = x_{m_n}$ for $m_n = 1$ for all n. The sequence of indices $\{m_n\}_{n=1}^{\infty}$ is not strictly increasing.

4) Since the problem does not say that the subsequences of $\{\sin \frac{n\pi}{4}\}_{n=1}^{\infty}$ chosen must converge to different limits, it suffices to choose one convergent subsequence and then five subsequences of this subsequence. (A subsequence of a subsequce is a subsequence of the original sequence; a subsequence of a convergent sequence converges.) Take for example:

$$\left\{ \sin \frac{4n\pi}{4} \right\}_{n=1}^{\infty} = \{\sin(n\pi)\}_{n=1}^{\infty} = 0, 0, 0. \ldots$$

and

$$\left\{ \sin \frac{8n\pi}{4} \right\}_{n=1}^{\infty}, \quad \left\{ \sin \frac{12n\pi}{4} \right\}_{n=1}^{\infty}, \quad \left\{ \sin \frac{16n\pi}{4} \right\}, \quad \left\{ \sin \frac{20n\pi}{4} \right\},$$

$$\left\{ \sin \frac{24n\pi}{4} \right\}_{n=1}^{\infty}.$$

5) Let $\{a_n\}_{n=1}^{\infty}$ be an unbounded sequence. By Problem 1, there exists a subsequence $\{a_{m_n}\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} a_{m_n} = +\infty$ or there exists a subsequence $\{a_{p_n}\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} a_{p_n} = -\infty$.

Easily we prove that if such $\{a_{m_n}\}_{n=1}^{\infty}$ exists, then $\lim_{n\to+\infty} a_{m_n} = 0$. (See H5, Pr. 2.). If such $\{a_{p_n}\}_{n=1}^{\infty}$ exists, then

$\lim_{n\to+\infty} a_{p_n} = 0$.

6) Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge to L. Then

$$\exists_{\varepsilon_0>0} \ \forall_{N\in\mathbb{N}} \ \exists_{n\geq N} \ |x_n - L| \geq \varepsilon_0. \qquad (2)$$

Take and fix ε_0 which satisfies (2). Applying (2) with $N=1$, choose $m_1 \in \mathbb{N}$ such that $|x_{m_1} - L| \geq \varepsilon_0$. Applying (2) with $N = m_1+1$ we can choose $m_2 \in \mathbb{N}$ such that

$$m_2 > m_1 \quad \text{and} \quad |x_{m_2} - L| \geq \varepsilon_0.$$

Applying (2) with $N = m_2+1$, choose $m_3 \in \mathbb{N}$ such that

$$m_3 > m_2 > m_1 \quad \text{and} \quad |x_{m_3} - L| \geq \varepsilon_0.$$

And so on, there exists a sequence m_n, $n=1,2,\ldots$ such that:

$$m_1 < m_2 < \ldots < m_n < m_{n+1} < \ldots \quad \text{and} \quad |x_{m_n} - L| \geq \varepsilon_0. \ (3)$$

(Of course, more precisely we are defining $\{m_n\}_{n=1}^{\infty}$ by induction as in Problem 2.) By (3), $\{x_{m_n}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$. Since $\{x_n\}_{n=1}^{\infty}$ was assumed bounded, $\{x_{m_n}\}_{n=1}^{\infty}$ is bounded. By Bolzano-Weierstrass theorem $\{x_{m_n}\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{m_{p_n}}\}_{n=1}^{\infty}$. A subsequence of a subsequence of $\{x_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$. Hence, $\{x_{m_{p_n}}\}_{n=1}^{\infty}$ is a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ and by (3)

$$|x_{m_{p_n}} - L| \geq \varepsilon_0 \text{ for all } n \in \mathbb{N}.$$

Thus, $\{x_{m_{p_n}}\}_{n=1}^{\infty}$ does not converge to L. Contradiction as every convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to L. Therefore, $\{x_n\}_{n=1}^{\infty}$ converges to L.