1) Suppose $L > M$. Let $\varepsilon = \frac{L-M}{2}$, $m = \frac{L+M}{2}$. Then the following conditions hold:

$$\varepsilon > 0 \;,\; L - \varepsilon = m, \quad M + \varepsilon = m. \qquad (1)$$

Let $N_1, N_2 \in \mathbb{N}$ be such that

$$|a_n - L| < \varepsilon \text{ for all } n \geq N_1 \text{ and } |b_n - M| < \varepsilon \text{ for all } n \geq N_2. \quad (2)$$

Take $N = \max\{N_0, N_1, N_2\}$. Then, by (2) and the assumptions, for all $n \geq N$ the following conditions are satisfied:

$$a_n \leq b_n \;,\; L - \varepsilon < a_n < L + \varepsilon \;,\; M - \varepsilon < b_n < M + \varepsilon.$$

The conditions together with (1) give that for $n \geq N$:

$$a_n \leq b_n \;,\quad a_n > m \;,\quad b_n < m.$$

Contradiction. Hence, $L \leq M$.

2) We use Def. 11.1. Let $M > 0$ be given. Then, $\frac{1}{M} > 0$. Since $b_n > 0$ and $\lim_{n \to +\infty} b_n = 0$, there exists $N \in \mathbb{N}$ such that

$$b_n < \frac{1}{M} \quad \text{for } n \geq N,$$

As $M > 0$ an $b_n > 0$, the latter inequality implies:

$$\frac{1}{b_n} > M \quad \text{for } n \geq N.$$

Thus, $\lim_{n \to +\infty} \frac{1}{b_n} = +\infty$.

3) Since $A \subseteq B$, any lower bound for B is a lower bound for A. In particular, $\inf B$ is a lower bound for A. Thus:

$$\inf B \leq \inf A$$

from the definition of $\inf A$. Similarly, any upper bound for B is an upper bound for A. In particular, $\sup B$ is an upper bound for A. Hence:

$$\sup B \geq \sup A.$$

4) No. Let $a_n = -1 - \frac{1}{n}$, $b_n = 1 + \frac{1}{n}$, $c_n = (-1)^n$ for $n = 1, 2, \ldots$

$\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are Cauchy sequences as they are convergent. $\{c_n\}_{n=1}^{\infty}$ is not a Cauchy sequence even though $a_n \leq c_n \leq b_n$.

5) Let $\varepsilon > 0$ be given. By Def 11.1, there exists $N \in \mathbb{N}$ such that

$$b_n < -\frac{1}{\varepsilon} \quad \text{for } n \geq N. \qquad (3)$$

Hence $b_n < 0$ and since $\varepsilon > 0$, (3) gives:

$$\varepsilon > -\frac{1}{b_n} = \left|\frac{1}{b_n}\right| \quad \text{for } n \geq N.$$

Therefore, $\lim\limits_{n \to +\infty} \frac{1}{b_n} = 0$.

6) Our assumptions imply that for all $n \in \mathbb{N}$:

$$a_1 \leq a_n \leq a_{n+1} < b_{n+1} \leq b_n \leq b_1. \qquad (4)$$

Hence, $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded, $\{b_n\}_{n=1}^{\infty}$ is decreasing and bounded. By Th 11.1 both sequences are convergent. Denote:

$$\lim_{n \to \infty} a_n = a, \quad \lim_{n \to +\infty} b_n = b.$$

By Problem 1 and (4), $a \leq b$. Also, since $a = \sup\{a_n : n \in \mathbb{N}\}$, $b = \inf\{b_n : n \in \mathbb{N}\}$, $a \geq a_n$, $b_n \geq b$ for all $n \in \mathbb{N}$. Hence,

$$[a, b] \subseteq [a_n, b_n] \text{ for all } n \in \mathbb{N} \qquad (5)$$

and $[a,b] \neq \emptyset$ as $a \leq b$. By (5) any point $z \in [a,b]$ belongs to all intervals $[a_n, b_n]$; that is, to the intersection $\bigcap_{n=1}^{\infty} [a_n, b_n]$. The first part is proved. To prove the second part assume additionally that

$$\lim_{n \to +\infty} (b_n - a_n) = 0 \qquad (6)$$

By Th. 10.2 $\lim\limits_{n \to +\infty} (b_n - a_n) = b - a = 0$. Hence, $b = a$ and $[a,b] = \{a\}$.

From what we proved above $z = a \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. Suppose $y \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Then $0 \leq |z - y| \leq b_n - a_n$ for all $n \in \mathbb{N}$. From (6), Th 10.1, and Remark 10.1, $|z - y| = 0$. Thus, $y = z$.