Homework 4 MTH 435 F05 - Solutions

1) For every $t \in T$ we have $t \leq s_1$. For every $v \in V$ we have $v \leq s_2$. Thus for every $t+v \in S$ we have $t+v \leq s_1+s_2$. Hence, s_1+s_2 is an upper bound for S. We shall use Th 7.3. to show that

$$s_1 + s_2 = \sup S. \qquad (1)$$

We have shown (a) of Th 7.3, now we'll show (b).
Let $\varepsilon > 0$ be given. Applying Th 7.3 to T and V, we obtain that there exist $t_0 \in T$ and $v_0 \in V$ such that

$$s_1 - \frac{\varepsilon}{2} < t_0 \leq s_1 \quad , \quad s_2 - \frac{\varepsilon}{2} < v_0 \leq s_2 . \qquad (2)$$

Indeed $s_1 = \sup T$, $s_2 = \sup V$, hence they both satisfy (b) of Th 7.3. Adding the inequalities (2), we obtain

$$(s_1+s_2) - \varepsilon < t_0 + v_0 \leq s_1 + s_2 .$$

Thus, $s_1 + s_2$ satisfies (b) of Th 7.3, and (1) holds. ∎

2) The inequality

$$|x| - |y| \leq |x-y| \qquad (3)$$

is equivalent to

$$|x| \leq |x-y| + |y|. \qquad (4)$$

Use the triangle inequality $|a+b| \leq |a| + |b|$ with $a = x-y$, $b = y$. (4) follows. Thus (3). ∎

3) We use Def 8.4. Let $\varepsilon > 0$ be given. Take N_ε such that $N_\varepsilon \in \mathbb{N}$ and $N_\varepsilon > \frac{1}{\sqrt{\varepsilon}}$. Then for every $n \geq N_\varepsilon$ we have

$$\frac{1}{n^2} < \varepsilon .$$

Thus for $n \geq N_\varepsilon$, $\left| 1 - \frac{1}{n^2} - 1 \right| = \left| \frac{1}{n^2} \right| = \frac{1}{n^2} < \varepsilon$. ∎

4) $\lim_{n\to+\infty} \frac{(-1)^n}{\sqrt{n}} = 0$. We prove it using Def 8.4. Let $\varepsilon > 0$ be given.
Let $N_\varepsilon \in \mathbb{N}$ be such that $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then for every $n \geq N_\varepsilon$ we have

$$\left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}} < \varepsilon. \quad \blacksquare$$

5) Since $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are bounded, there exist $K, M \in \mathbb{R}$ such that

$$|a_n| \leq K, \quad |b_n| \leq M \quad \text{for all } n \in \mathbb{N}. \qquad (5)$$

Using the Triangle Inequality, (5) and Remark 9.1, we obtain for all $n \in \mathbb{N}$:

$$|a_n + b_n| \leq |a_n| + |b_n| \leq K + M,$$

$$|a_n - b_n| = |a_n + (-b_n)| \leq |a_n| + |b_n| \leq K + M,$$

$$|a_n \cdot b_n| = |a_n| \cdot |b_n| \leq K \cdot M.$$

(By (5), $K \geq 0$ and $M \geq 0$. Also, $|b_n| = |-b_n|$ by Def 8.1).
Hence, $\{a_n + b_n\}$, $\{a_n - b_n\}$, $\{a_n \cdot b_n\}$ are all bounded as $K+M, K \cdot M \in \mathbb{R}$

6) We use Def 8.4. Let $\varepsilon > 0$ be given. Since $\lim_{n\to+\infty} a_n = L$, there exists, $N_1 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for } n \geq N_1. \qquad (6)$$

Let $N = \max\{N_1, N_0\}$. Then for any $n \geq N$, (6) holds as well as $c_n = a_n$. Thus:

$$|c_n - L| < \varepsilon \text{ for } n \geq N.$$

We conclude $\lim_{n\to+\infty} c_n = L$. $\blacksquare$

Observe that the latter theorem means that convergence and the limit of a given sequence does not depend on any finite member of its elements. In other words, you can change any finite number of elements of a sequence without affecting its limit or convergence.