MTH 435 Homework 3 - Solutions F 05

1) Proof very similar to Proof of Th 7.3 by obvious modifications and Def 5.8.

2) As $B \subseteq (1,2)$, $b < 2$ for all $b \in B$, so 2 is an upper bound for B.
To prove that $2 = \sup B$ we use Th 7.3 and Th 7.1. Let $\varepsilon > 0$ be given.
By Th 7.1, there exists a rational number q such that

$$\max\{2-\varepsilon, 1\} < q < 2.$$

Then $q \in B$. As ε was chosen arbitrarily, $s = 2$ satisfies both conditions (a), (b) of Th 7.3. Hence, $2 = \sup B$.

3) Prove very similar to 2) using Th 7.4 and Th 7.2.

4) Since $A \neq \emptyset$, there exists $a \in A$. By Def 5.7 and Def 5.8 we have then:

$$\inf A \leq a \leq \sup A.$$

Thus, $\inf A \leq \sup A$. The equality hold for any singleton, that is, a one-element set. For example, $A = \{0\}$.

5) "$\Rightarrow$" Assume that A has the maximum, $a_0 = \max A$. By Def H3.1, a_0 is an upper bound for A. Since $a_0 \in A$ no smaller number can be an upper bound for A. Hence, $a_0 = \sup A$ and $\sup A \in A$.
"$\Leftarrow$" Let $s = \sup A$. Assume that $s \in A$. Since, s is an upper bound we have $a \leq s$ for all $a \in A$. Thus, be Def H3.1, s is the maximum of A.

6) To prove that S in dense, take any $a, b \in \mathbb{R}$, $a < b$. Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ and by Th 7.1, there exists $q \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}.$$

Since $q \in \mathbb{Q}$, $q = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Hence,

$$a < \frac{m\sqrt{2}}{n} < b.$$

Since $\frac{m\sqrt{2}}{n} \in S$, the claim is proved.