1) Prove by induction. For $n=1$, the formula becomes:

$$1=1$$

so it is true. Let $n\in\mathbb{N}$ be fixed. Assume that

(IA) $$1+3+\ldots+(2n-1)=n^2$$

We have to prove that:

$$1+3+\ldots+(2n-1)+(2(n+1)-1)=(n+1)^2 \quad (1)$$

By the inductive assumption (IA), the left-hand side of (1) becomes:

$$1+3+\ldots+(2n-1)+2n+1=n^2+2n+1.$$

As $(n+1)^2=n^2+2n+1$, (1) holds. Hence, for all $n\in\mathbb{N}$, $1+3+\ldots+(2n-1)=n^2$. ∎

2) (b) Let $a\in\mathbb{R}$, $a>-1$. We have to prove that

$$(1+a)^n \geq 1+na \quad \text{for all } n\in\mathbb{N}. \quad (2)$$

We use the finite induction principle.

For $n=1$ the formula (2) becomes

$$1+a \geq 1+a,$$

so it is true.

Let $k\in\mathbb{N}$ be given. Assume that

(IA) $$(1+a)^k \geq 1+ka.$$

We have to prove that

$$(1+a)^{k+1} \geq 1+(k+1)a. \quad (3)$$

Indeed, the (IA) and $a>-1$ imply:

$$(1+a)^{k+1}=(1+a)^k(1+a)\geq(1+ka)(1+a)=$$

$$=1+ka+a+ka^2=1+(k+1)a+ka^2\geq$$

$$\geq 1+(k+1)a.$$

The latter inequality holds as $ka^2\geq 0$. Thus (3) holds, which concludes the proof. ∎

2)(a) We have to prove for a fixed $a \in \mathbb{R}$, $a \neq -1$:

$$1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a} \quad \text{for all } n \in \mathbb{N}. \quad (3)$$

We use induction.

For $n = 1$ the formula becomes:

$$1 + a = \frac{1-a^2}{1-a}.$$

Since $1 - a^2 = (1-a)(1+a)$, the formula is true.

Let $k \in \mathbb{N}$ be given. Assume:

$$\text{(IA)} \quad 1 + a + \ldots + a^k = \frac{1-a^{k+1}}{1-a}.$$

We have to prove that

$$1 + a + \ldots + a^k + a^{k+1} = \frac{1-a^{(k+1)+1}}{1-a}, \quad (4).$$

Indeed, by (I A):

$$1 + a + \ldots + a^k + a^{k+1} = (1 + a + \ldots + a^k) + a^{k+1} =$$

$$= \frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}.$$

Thus (4) holds, and, by induction, (3) holds. ∎

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3) For every $ka \in B$ we have $ka \leq ks$ as $k > 0$ and $a \leq s$ for all $a \in A$. Thus ks is an upper bound for B. Suppose $p < ks$ is an upper bound for B. Then, $ka \leq p < ks$ for all $a \in A$. Thus, for all $a \in A$:

$$a \leq \frac{1}{k} p < s.$$

Contradiction as $s = \sup A$. If $k < 0$, $ks = \inf B$. Prove it yourself. For $k = 0$, $B = \{0\}$, $ks = 0$. Hence,

$ks = \sup B = \inf B$.

4) Since $s_1 = \sup A$, $s_2 = \sup B$, we have:

$$a_n \leq s_1 \quad , \quad b_n \leq s_2 \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$a_n + b_n \leq s_1 + s_2 \quad \text{for all } n \in \mathbb{N}.$$

The latter shows that $s_1 + s_2$ is an upper bound for C. $s_1 + s_2$ need not be the supremum of C. Indeed, let:

$$A = \{1, 0, 1, 0, 1, 0, \dots\} \quad , \quad B = \{0, 1, 0, 1, 0, 1, \dots\}.$$

Then $s_1 = s_2 = 1$, $s_1 + s_2 = 2$, and $C = \{1, 1, 1, 1, 1, \dots\}$. Hence, $\sup C = 1 < s_1 + s_2$.

5) As $1 \in A$ no upper bound for A can be less than 1. 1 is an upper bound for A as $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$. Thus, $\sup A = 1$.

For all $n \in \mathbb{N}$, $\frac{1}{n} > 0$. Thus, 0 is a lower bound for A. Suppose $p > 0$. By AOP, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < p$. Hence, p cannot be a lower bound for A. We conclude that $0 = \inf A$.

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