

1) Let $x_0 \neq 0$. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence such that $q_n \in \mathbb{Q}$ for $n=1, 2, \dots$ and $q_n \rightarrow x_0$ as $n \rightarrow +\infty$. Then, $f(q_n) = q_n \rightarrow x_0$ as $n \rightarrow +\infty$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence such that $r_n \in \mathbb{R} \setminus \mathbb{Q}$ for $n=1, 2, \dots$, $r_n \rightarrow x_0$ as $n \rightarrow +\infty$. Then $f(r_n) = 0 \rightarrow 0$ as $n \rightarrow +\infty$. As $x_0 \neq 0$, by Heine's characterization of the limit, $\lim_{x \rightarrow x_0} f(x)$ does not exist. Hence, f is not continuous and thus not differentiable at x_0 . (The existence of the sequences $\{q_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ as required above follows easily from the density of rationals and irrationals. Indeed, for each $n=1, 2, \dots$ define q_n to be any number in $(x_0, x_0 + \frac{1}{n}) \cap \mathbb{Q}$, r_n to be any number in $(x_0, x_0 + \frac{1}{n}) \setminus \mathbb{Q}$.)

Let $x_0 = 0$. f is continuous at x_0 . Indeed, let $\epsilon > 0$ be given. Take $\delta = \epsilon$. Then:

$$|f(x) - f(x_0)| = \begin{cases} |x|, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$$

Hence,

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x| < \delta.$$

f is not differentiable at $x_0 = 0$. Indeed, the limit of the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

as $x \rightarrow x_0$ does not exist. (Proof similar as above from Heine's characterization of the limit.)

$$2) \quad f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{xg(x)}{x} = \lim_{x \rightarrow 0} g(x).$$

The latter limit exists and is equal to $g(0)$ as g is continuous at 0. Hence, f is differentiable at $x=0$ and $f'(0) = g(0)$.

3) Since g is bounded in $[-1, 1]$, there exists a constant $K > 0$ such that

$$|g(x)| \leq K \quad \text{for all } x \in [-1, 1]. \quad (1)$$

To prove differentiability of f at $x=0$, examine

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 g(x)}{x} = \lim_{x \rightarrow 0} x g(x).$$

The latter limit exists and is equal to 0. Indeed, take any $\epsilon > 0$.

Take $\delta = \frac{\epsilon}{K}$. Then:

$$|x g(x) - 0| < \epsilon \quad \text{whenever } |x| < \delta.$$

Indeed, by (1):

$$|x g(x)| = |x| |g(x)| \leq |x| \cdot K < \epsilon \quad \text{if } |x| < \delta.$$

Hence, f is differentiable at 0 and $f'(0) = 0$.

4) Consider the function $f(x) = x - \sin x$. The function is continuous in $(-\infty, +\infty)$ from theorems given in class. Observe that, $f(0) = 0 < c$ as c is positive. Observe next that $f(c+2) = c+2 - \sin(c+2) \geq c+1 > c$. By the IVT, there exists a point $k \in (0, c+2)$ such that $f(k) = c$.

5) We have to show that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in (-\infty, +\infty) \quad (|x-y| < \delta \Rightarrow |(f(x)+g(x)) - (f(y)+g(y))| < \epsilon). \quad (2)$$

Let $\epsilon > 0$ be fixed. As f is uniformly continuous in $(-\infty, +\infty)$ (and $\frac{\epsilon}{2}$ is a positive number) there exists $\delta_1 > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{whenever } |x-y| < \delta_1, \quad x, y \in (-\infty, +\infty). \quad (3)$$

As g is uniformly continuous in $(-\infty, +\infty)$, there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \frac{\epsilon}{2} \quad \text{whenever } |x-y| < \delta_2, \quad x, y \in (-\infty, +\infty). \quad (4)$$

Let $\delta = \min\{\delta_1, \delta_2\}$. From (3) and (4):

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - g(y)| < \frac{\epsilon}{2} \quad \text{whenever } |x-y| < \delta. \quad (5)$$

Observe that from (5):

$| (f(x)+g(x)) - (f(y)+g(y)) | \leq |f(x)-f(y)| + |g(x)-g(y)| < \epsilon$
whenever $|x-y| < \delta$. Since ϵ was arbitrary (2) holds. Thus, $f+g$ is uniformly continuous in $(-\infty, +\infty)$.

6) By contradiction. Suppose f has two distinct fixed points a, b . Without any loss of generality, we may assume $a < b$. Since f is differentiable in $(-\infty, +\infty)$, f is continuous in $(-\infty, +\infty)$ and thus continuous in $[a, b]$. Also, f is differentiable in (a, b) as $(a, b) \subseteq (-\infty, +\infty)$. By the MVT, there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But $f(b) = b$ and $f(a) = a$ as a, b are fixed points. Thus $f'(c) = 1$. Contradiction, as $f'(x) < 1$ for all $x \in \mathbb{R}$.