Below are a few solutions to Practice Test II problems.

Sec. 16.2

3. We evaluate the inside integral first:

$$\int_0^2 (x^2y)\,dy = \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=0}^{y=2} = 2x^2.$$

Therefore, we have

$$\int_0^1\int_0^2 (x^2y)\,dydx = \int_0^1 (2x^2)\,dx = \left(\frac{2x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}.$$

9. The line connecting $(1,0)$ and $(4,1)$ is

$$y = \frac{1}{3}(x-1)$$

So the integral is

$$\int_1^4\int_{(x-1)/3}^2 f\,dy\,dx$$

15. The region of integration ranges from $x=0$ to $x=3$ and from $y=0$ to $y=2x$, as shown in Figure 16.11. To evaluate the integral, we evaluate the inside integral first:

$$\int_0^{2x} (x^2+y^2)\,dy = \left(x^2y+\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=2x} = x^2(2x)+\frac{(2x)^3}{3} = 2x^3+\frac{8x^3}{3} = \frac{14}{3}x^3.$$

Therefore, we have

$$\int_0^3\int_0^{2x} (x^2+y^2)\,dydx = \int_0^3 \left(\frac{14}{3}x^3\right)\,dx = \left(\frac{14}{12}x^4\right)\Bigg|_0^3 = 94.5.$$

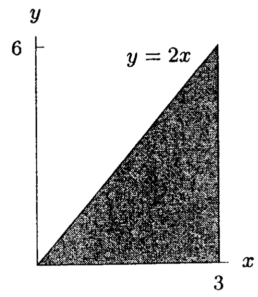

Figure 16.11

27. The function $\sin(x^2)$ has no elementary antiderivative, so we try integrating with respect to y first. The region of integration is shown in Figure 16.17. Changing the order of integration, we get

$$\begin{aligned}\int_0^1\int_y^1 \sin(x^2)\,dx\,dy &= \int_0^1\int_0^x \sin(x^2)\,dy\,dx\\ &= \int_0^1 \sin(x^2)\cdot y\Big|_0^x\,dx\\ &= \int_0^1 \sin(x^2)\cdot x\,dx\\ &= -\frac{\cos(x^2)}{2}\Bigg|_0^1\\ &= -\frac{\cos 1}{2}+\frac{1}{2} = \frac{1}{2}(1-\cos 1) = 0.23.\end{aligned}$$

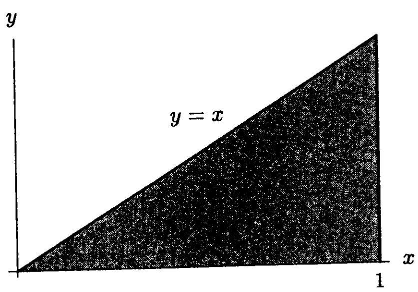

Figure 16.17

15.2 # 21

21. We calculate the partial derivatives and set them to zero.

$$\frac{\partial\,(\text{range})}{\partial t} = -10t - 6h + 400 = 0$$
$$\frac{\partial\,(\text{range})}{\partial h} = -6t - 6h + 300 = 0.$$

$$10t + 6h = 400$$
$$6t + 6h = 300$$

solving we obtain

$$4t = 100$$

so

$$t = 25$$

Solving for h, we obtain $6h = 150$, yielding $h = 25$. Since the range is quadratic in h and t, the second derivative test tells us this is a local and global maximum. So the optimal conditions are $h = 25\%$ humidity and $t = 25°$C.

14.7

21. (a) $f_x(P) < 0$ because f decreases as you go to the right.
(b) $f_y(P) = 0$ because f does not change as you go up.
(c) $f_{xx}(P) < 0$ because f_x decreases as you go to the right (f_x changes from a small negative number to a large negative number).
(d) $f_{yy}(P) = 0$ because f_y does not change as you go up.
(e) $f_{xy}(P) = 0$ because f_x does not change as you go up.

29. We have $f(1,0) = 1$ and the relevant derivatives are:

$$f_x = \frac{1}{2}(x+2y)^{-1/2} \quad \text{so} \quad f_x(1,0) = \frac{1}{2}$$
$$f_y = (x+2y)^{-1/2} \quad \text{so} \quad f_y(1,0) = 1$$
$$f_{xx} = -\frac{1}{4}(x+2y)^{-3/2} \quad \text{so} \quad f_{xx}(1,0) = -\frac{1}{4}$$
$$f_{xy} = -\frac{1}{2}(x+2y)^{-3/2} \quad \text{so} \quad f_{xy}(1,0) = -\frac{1}{2}$$
$$f_{yy} = -(x+2y)^{-3/2} \quad \text{so} \quad f_{yy}(1,0) = -1\,.$$

Thus the linear approximation, $L(x,y)$ to $f(x,y)$ at $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + \frac{1}{2}(x-1) + y\,. \end{aligned}$$

The quadratic approximation, $Q(x,y)$ to $f(x,y)$ near $(1,0)$, is given by:

$$\begin{aligned} f(x,y) \approx Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\quad + f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= 1 + \frac{1}{2}(x-1) + y - \frac{1}{8}(x-1)^2 - \frac{1}{2}(x-1)y - \frac{1}{2}y^2\,. \end{aligned}$$

The values of the approximations are

$$L(0.9, 0.2) = 1 - 0.05 + 0.2 = 1.15$$
$$Q(0.9, 0.2) = 1 - 0.05 + 0.2 - 0.00125 + 0.01 - 0.02 = 1.13875$$

and the exact value is

$$f(0.9, 0.2) = \sqrt{1.3} \approx 1.14018.$$

Observe that the quadratic approximation is closer to the exact value.

14.6

3

18. The voltage at any time t is given by $V = IR$ where R is the resistance for the whole circuit. (In this case $R = R_1R_2/(R_1+R_2)$.) So the rate at which the voltage is changing is

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dI}{dt}R + I\frac{dR}{dt}\\
&= \frac{dI}{dt}R + I\left(\frac{\partial R}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial R}{\partial R_2}\frac{dR_2}{dt}\right)\\
&= \frac{dI}{dt}R + I\left(\frac{R_2^2}{(R_1+R^2)^2}\frac{dR_1}{dt} + \frac{R_1^2}{(R_1+R_2)^2}\frac{dR_2}{dt}\right)\\
&= 0.01\left(\frac{15}{8}\right) + 2\left(\frac{25}{64}(0.5) + \frac{9}{64}(-0.1)\right)\\
&= 0.3812.
\end{aligned}$$

So the voltage is increasing by 0.3812 volts/sec.

14.5

17. First, we check that $(-1)^2 - (1)^2 + 2^2 = 4$. Then let $f(x,y,z) = x^2 - y^2 + z^2$ so that the given surface is the level surface $f(x,y,z) = 4$. Since $f_x = 2x$, $f_y = -2y$, and $f_z = 2z$, we have grad $f(-1,1,2) = -2\vec{i} - 2\vec{j} + 4\vec{k}$. Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,1,2)$ is $\vec{n} = -2\vec{i} - 2\vec{j} + 4\vec{k}$. Thus an equation for the tangent plane is

$$-2(x+1) - 2(y-1) + 4(z-2) = 0.$$

24. **(a)** We have $\nabla G = (2x-5y)\vec{i} + (-5x+2yz)\vec{j} + (y^2)\vec{k}$, so $\nabla G(1,2,3) = -8\vec{i} + 7\vec{j} + 4\vec{k}$. The rate of change is given by the directional derivative in the direction $\vec{v}$:

$$\begin{aligned}
\text{Rate of change in density} &= \nabla G \cdot \frac{\vec{v}}{\|\vec{v}\|} = (-8\vec{i} + 7\vec{j} + 4\vec{k}) \cdot \frac{(2\vec{i} + \vec{j} - 4\vec{k})}{\sqrt{21}}\\
&= \frac{-16+7-16}{\sqrt{21}} = \frac{-25}{\sqrt{21}} \approx -5.455.
\end{aligned}$$

(b) The direction of maximum rate of change is $\nabla G(1,2,3) = -8\vec{i} + 7\vec{j} + 4\vec{k}$.

(c) The maximum rate of change is $\|\nabla G(1,2,3)\| = \sqrt{(-8)^2 + 7^2 + 4^2} = \sqrt{129} \approx 11.36$.

14.4

56. Assume that the x-axis points east and the y-axis points north. We are given that $\|\nabla f\| = 5$ and that ∇f is in the direction $\vec{i} + \vec{j}$. Since $\|\vec{i} + \vec{j}\| = \sqrt{2}$ and ∇f is a multiple of $\vec{i} + \vec{j}$, we have

$$\nabla f = \frac{5}{\sqrt{2}}(\vec{i} + \vec{j}).$$

The rate of change toward the north is the directional derivative in direction $\vec{j}$, which is

$$\nabla f \cdot \vec{j} = \frac{5}{\sqrt{2}}(\vec{i} + \vec{j}) \cdot \vec{j} = \frac{5}{\sqrt{2}}.$$

59. Directional derivative = $\nabla f \cdot \vec{u}$, where $\vec{u}$ = unit vector. If we move from $(4,5)$ to $(5,6)$, we move in the direction $\vec{i} + \vec{j}$ so $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$. So,

$$\nabla f \cdot \vec{u} = f_x\left(\frac{1}{\sqrt{2}}\right) + f_y\left(\frac{1}{\sqrt{2}}\right) = 2.$$

Similarly, if we move from $(4,5)$ to $(6,6)$, the direction is $2\vec{i} + \vec{j}$ so $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$. So

$$\nabla f \cdot \vec{u} = f_x\left(\frac{2}{\sqrt{5}}\right) + f_y\left(\frac{1}{\sqrt{5}}\right) = 3.$$

Solving the system of equations for f_x and f_y

$$\begin{aligned}
f_x + f_y &= 2\sqrt{2}\\
2f_x + f_y &= 3\sqrt{5}
\end{aligned}$$

gives

$$\begin{aligned}
f_x &= 3\sqrt{5} - 2\sqrt{2}\\
f_y &= 4\sqrt{2} - 3\sqrt{5}.
\end{aligned}$$

Thus at $(4,5)$,

$$\nabla f = (3\sqrt{5} - 2\sqrt{2})\vec{i} + (4\sqrt{2} - 3\sqrt{5})\vec{j}.$$

14. We have

$$f_x(3,1) = \left.\frac{\partial f}{\partial x}\right|_{(3,1)} = 2xy|_{(3,1)} = 6,$$

and

$$f_y(3,1) = \left.\frac{\partial f}{\partial y}\right|_{(3,1)} = x^2|_{(3,1)} = 9.$$

Also $f(3,1) = 9$. So the local linearization is,

$$z = 9 + 6(x-3) + 9(y-1).$$

15. Since $f_x(x,y) = \frac{x}{\sqrt{x^2+y^3}}$ and $f_y(x,y) = \frac{3y^2}{2\sqrt{x^2+y^3}}$,
$f_x(1,2) = \frac{1}{\sqrt{1^2+2^3}} = \frac{1}{3}$ and $f_y(1,2) = \frac{3\cdot 2^2}{2\sqrt{1^2+2^3}} = 2$.
Thus the differential at the point $(1,2)$ is

$$df = df(1,2) = f_x(1,2)dx + f_y(1,2)dy = \frac{1}{3}dx + 2dy.$$

Using the differential at the point $(1,2)$, we can estimate $f(1.04, 1.98)$. Since

$$\Delta f \approx f_x(1,2)\Delta x + f_y(1,2)\Delta y$$

where $\Delta f = f(1.04, 1.98) - f(1,2)$ and $\Delta x = 1.04 - 1$ and $\Delta y = 1.98 - 2$, we have

$$\begin{aligned} f(1.04, 1.98) &\approx f(1,2) + f_x(1,2)(1.04-1) + f_y(1,2)(1.98-2) \\ &= \sqrt{1^2+2^3} + \frac{0.04}{3} - 2(0.02) \approx 2.973. \end{aligned}$$

16. Local linearization gives us the approximation

$$\begin{aligned} T(x,y) &\approx T(2,1) + T_x(2,1)(x-2) + T_y(2,1)(y-1) \\ T(x,y) &\approx 135 + 16(x-2) - 15(y-1). \end{aligned}$$

Thus,

$$T(2.04, 0.97) \approx 135 + 16(2.04-2) - 15(0.97-1) = 136.09^\circ\text{C}.$$