New results in t-tone coloring of graphs

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Abstract

A t-tone k-coloring of G assigns to each vertex of G a set of t colors from $\{1,\dots,k\}$ so that vertices at distance d share fewer than d common colors. The *t-tone chromatic number* of G, denoted $\tau_t(G)$, is the minimum k such that G has a t-tone k-coloring. Bickle and Phillips showed that always $\tau_2(G) \le [\Delta(G)]^2 + \Delta(G)$, but conjectured that in fact $\tau_2(G) \le 2\Delta(G) + 2$; we confirm this conjecture when $\Delta(G) \le 3$ and also show that always $\tau_2(G) \le \lceil (2+\sqrt{2})\Delta(G) \rceil$. For general t we prove that $\tau_t(G) \le (t^2+t)\Delta(G)$. Finally, for each $t \ge 2$ we show that there exist constants c_1 and c_2 such that for every tree T we have $c_1\sqrt{\Delta(T)} \le \tau_t(T) \le c_2\sqrt{\Delta(T)}$.

1 Introduction

In standard vertex coloring, we give colors to the vertices of a graph so that adjacent vertices get distinct colors. This well-studied notion has given rise to many variants. Several of these variants place restrictions on the colors of vertices that are near each other, but not necessarily adjacent. In a *distance-k coloring*, any vertices within distance k of each other must receive distinct colors. Sometimes we impose strong restrictions on the colors of adjacent vertices, and weaker restrictions on vertices at greater distance; for example, in an *$L(2,1)$-labeling* [5] each vertex receives a nonnegative integer as its label, such that the labels on adjacent vertices differ by at least 2 and those on vertices at distance 2 differ by at least 1. Another variant, *set coloring* (also known as *n-tuple coloring*), assigns a set of colors to each vertex, with the restriction that adjacent vertices receive disjoint sets; see [3, 6, 7].

The notion of *t-tone coloring* combines and extends these ideas. Intuitively, a *t-tone k-coloring* of G assigns to each vertex of G a set of t colors from $\{1,\dots,k\}$ so that vertices at distance d share fewer than d common colors. This notion is especially appealing when $t = 2$. In this case, each vertex receives a set of two colors; adjacent vertices receive disjoint sets

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and vertices at distance 2 receive distinct sets. The concept of t-tone coloring was introduced by G. Chartrand and initially studied in a research group directed by P. Zhang, consisting of Fonger, Goss, Phillips, and Segroves [4]; additional results due to Bickle and Phillips appear in [2]. The t-tone chromatic number of random graphs was studied by Bal, Bennett, Dudek, and Frieze [1].

Before giving a formal definition, we first establish some basic notation and terminology. We write $[k]$ as shorthand for $\{1, \dots, k\}$ and denote by $\binom{[k]}{t}$ the family of t-element subsets of $[k]$. We denote the distance between vertices u and v by $d(u, v)$. Vertices u and v are *neighbors* if $d(u, v) = 1$ and *second-neighbors* if $d(u, v) = 2$.

Definition 1.1 *[4] Let G be a graph and t a positive integer. A* t-tone k-coloring *of G is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v. A graph that has a t-tone k-coloring is* t-tone k-colorable*. The* t-tone chromatic number *of G, denoted $\tau_t(G)$, is the minimum k such that G is t-tone k-colorable.*

Given a t-tone coloring f of G, we call $f(v)$ the *label* of v and the elements of $[k]$ *colors*. When the meaning is clear, we omit set notation from labels; that is, we denote the label $\{a, b\}$ by ab. Note that for each t, the parameter τ_t is monotone: when H is a subgraph of G, every t-tone k-coloring of G restricts to a t-tone k-coloring of H, so $\tau_t(H) \le \tau_t(G)$.

Fonger, Goss, Phillips, and Segroves [4] established several basic results on t-tone coloring, some of which focused on the relationship between τ_2 and other graph parameters. By looking at proper colorings of the graph G^2, they proved that $\tau_2(G) \le \chi(G^2) + \chi(G)$. In the case where $\chi(G) = \Delta(G) + 1$, Bickle and Phillips [2] obtained the slightly stronger bound $\tau_2(G) \le [\Delta(G)]^2 + \Delta(G)$ (valid when $\Delta(G) > 1$). However, they conjectured that this bound is far from tight:

Conjecture 1.2 *[2] If G is a graph with maximum degree r, then $\tau_2(G) \le 2r + 2$. If $r \ge 3$, then equality holds only when G contains K_{r+1}.*

Bickle and Phillips established this conjecture for $r = 2$. When G is 3-regular, they posed the following stronger conjecture:

Conjecture 1.3 *[2] If G is a 3-regular graph, then:*

(a) $\tau_2(G) \le 8$;

(b) $\tau_2(G) \le 7$ *when G does not contain K_4;*

(c) $\tau_2(G) \le 6$ *when G does not contain $K_4 - e$.*

Since they also characterized all 2-tone 5-colorable 3-regular graphs, this conjecture would yield a complete characterization of the 2-tone chromatic numbers of 3-regular graphs.

In Section 2, we focus on 2-tone colorings, with an eye toward proving Conjectures 1.2 and 1.3. As progress toward Conjecture 1.2, we give a short proof that always $\tau_2(G) \le \lceil (2 + \sqrt{2})\Delta(G) \rceil$. Simple modifications of this argument yield better bounds when G is bipartite or chordal. We next refute part (c) of Conjecture 1.3 by showing that the Heawood graph has 2-tone chromatic number 7. Finally, our main result in Section 2 confirms part (a) of Conjecture 1.3:

Theorem 1 *If G is a graph with $\Delta(G) \le 3$, then $\tau_2(G) \le 8$.*

In Section 3, we consider t-tone colorings for general t. Our main result is:

Theorem 2 *For each t there exists a constant $c = c(t)$ such that $\tau_t(T) \le c\sqrt{\Delta(T)}$ whenever T is a nontrivial tree, and this bound is asymptotically tight.*

For general graphs, our best bound is $\tau_t(G) \le (t^2 + t)\Delta(G)$. This result implies that, for fixed $\Delta(G)$, we have $\tau_t(G) \le ct^2$ for some constant c. The asymptotics of this bound are near-optimal with respect to t, since for each $r \ge 3$ there exist a constant c and graphs G_t such that $\Delta(G_t) = r$ and $\tau_t(G_t) \ge ct^2/\lg t$. Finally, when G has degeneracy at most k, we prove $\tau_t(G) \le kt + kt^2[\Delta(G)]^{1-1/t}$.

2 2-tone Coloring

In this section we focus on 2-tone coloring. We first attack Conjecture 1.2. It was shown in [2] that always $\tau_2(G) \le [\Delta(G)]^2 + \Delta(G)$; we improve this result by giving an upper bound on $\tau_2(G)$ that is linear in $\Delta(G)$, rather than quadratic. This proof—along with several others throughout the paper—proceeds by building a t-tone coloring of a graph iteratively, coloring one vertex at a time.

Definition 2.1 *A* partial t-tone k-coloring *of a graph G is a function $f : S \to \binom{[k]}{t}$, with $S \subseteq V(G)$, such that $|f(u) \cap f(v)| < d(u, v)$ whenever $u, v \in S$. Vertices not in S are* uncolored. *An* extension *of f to an uncolored vertex v is a partial coloring f' that assigns a label to v but otherwise agrees with f.*

It is important to note that a t-tone k-coloring of a subgraph H of G need not be a partial t-tone k-coloring of G, since the distance between two vertices may be smaller in G than in H.

Theorem 2.2 *For every nonempty graph G, we have $\tau_2(G) \le \lceil (2 + \sqrt{2})\Delta(G) \rceil$.*

Proof. Let $k = \lceil (2 + \sqrt{2})\Delta(G) \rceil$ and let $V(G) = \{v_1, \dots, v_n\}$. Starting with all vertices uncolored, we extend our partial coloring to $v_1, v_2, \dots, v_n$ in order. When extending to v_i, we need only enforce two constraints. First, the label on v_i cannot contain any color appearing on v_i's neighbors; there remain at least $\lceil \sqrt{2}\Delta(G) \rceil$ other colors, so at least $\binom{\sqrt{2}\Delta(G)}{2}$ labels are available. Next, the label on v_i cannot appear on any second-neighbor of v_i; this condition forbids at most $\Delta(G)(\Delta(G) - 1)$ labels. Since $\binom{\sqrt{2}\Delta(G)}{2} > \Delta(G)(\Delta(G) - 1)$, some label remains for use on v_i. ∎

Similar approaches yield tighter bounds on $\tau_2(G)$ for bipartite graphs and chordal graphs.

Proposition 2.3 *If G is a nonempty bipartite graph, then $\tau_2(G) \le 2\lceil \sqrt{2}\Delta(G) \rceil$.*

Proof. A *palette* is a set of colors; we construct a 2-tone coloring of G using two disjoint palettes, each of size $\lceil\sqrt{2}\Delta(G)\rceil$. We assign each partite set its own palette and color the vertices in each set using only colors from its palette. Since adjacent vertices are assured disjoint labels, it suffices to ensure that vertices at distance 2 receive distinct labels.

We color each partite set independently. Within a partite set, we order the vertices arbitrarily and color iteratively. Each vertex v has at most $\Delta(G)(\Delta(G)-1)$ second-neighbors. Since each palette admits $\binom{\sqrt{2}\Delta(G)}{2}$ labels, we may always extend a partial coloring to v. ∎

A *simplicial elimination ordering* of a graph G is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that the later neighbors of each vertex form a clique; it is well-known that chordal graphs are precisely those graphs having simplicial elimination orderings.

Proposition 2.4 *If G is a nonempty chordal graph, then $\tau_2(G) \leq \lceil(1+\sqrt{6}/2)\Delta(G)\rceil + 1$.*

Proof. Let $k = \lceil(1+\sqrt{6}/2)\Delta(G)\rceil + 1$. Let $v_1, \ldots, v_n$ be the reverse of a simplicial elimination ordering of G; note that, for each i, the earlier neighbors of v_i form a clique. We construct a 2-tone k-coloring of G by coloring iteratively with respect to this ordering.

Suppose we want to color v_i. Let S be the set of earlier neighbors of v_i, and let $d = |S|$. If v_j is a later neighbor of v_i, then by our choice of ordering, all earlier neighbors of v_j are adjacent to v_i. Hence every earlier second-neighbor of v_i is adjacent to some vertex in S. Each vertex in S is adjacent to v_i itself along with the other $d-1$ vertices of S. Hence v_i has at most $d(\Delta(G)-d)$ earlier second-neighbors (note that $d \leq \Delta$).

Exactly $2d$ colors appear on S, so $k-2d$ colors remain. We have $\binom{k-2d}{2}$ labels using these colors, so we need $\binom{k-2d}{2} > d(\Delta(G)-d)$. It suffices to ensure that $(k-2d-1)^2 > 2d(\Delta(G)-d)$, which simplifies to $k > \sqrt{2d(\Delta(G)-d)} + 2d + 1$; maximizing the right side of this inequality with respect to d yields $k \geq \lceil(1+\sqrt{6}/2)\Delta(G)\rceil + 1$. ∎

Proposition 2.5 *For every $\epsilon > 0$, there exists an r_0 such that whenever $r > r_0$, if G is a chordal graph with maximum degree r, then $\tau_2(G) \leq (2+\epsilon)r$.*

Proof. Let G be a chordal graph with maximum degree r. Král [8] showed that, for some constant c, the graph G^2 is $cr^{3/2}$-degenerate. Thus, there is some ordering $v_1, \ldots, v_n$ of $V(G)$ such that each vertex has at most $cr^{3/2}$ earlier second-neighbors. Let us color iteratively with respect to this ordering using $k+2r$ colors, for some k to be specified later. When coloring v_i, as many as $2r$ colors may appear on its neighbors; at least k other colors remain. Thus we may color v_i so long as it has fewer than $\binom{k}{2}$ earlier second-neighbors; taking $k \geq \sqrt{2c}r^{3/4}+1$ suffices. Hence $\tau_2(G) \leq 2r + \sqrt{2c}r^{3/4} + 1$, from which the claim follows. ∎

We next turn our attention to 3-regular graphs and Conjecture 1.3. Later in this section, we prove part (a) of Conjecture 1.3 by showing that $\tau_2(G) \leq 8$ whenever $\Delta(G) \leq 3$; first we disprove part (c) by showing that the Heawood Graph, which has girth 6, has 2-tone chromatic number 7.

Theorem 2.6 *The Heawood Graph is not 2-tone 6-colorable.*

Proof. Let G denote the Heawood Graph. Recall that G is the incidence graph of the Fano Plane; thus it is bipartite, and every two distinct vertices in the same partite set have exactly one common neighbor (and hence lie at distance 2). Call a 2-tone 6-coloring of G a *good coloring*. For distinct colors a, b, c, d, call the set of labels $\{ab, cd, ac, bd\}$ a *complementary pair*. For distinct colors a, b, c, d, e, f, call the set of labels $\{ab, cd, ef\}$ a *disjoint triple*. Let A and B denote the partite sets of G.

We prove three claims: **(1)** No good coloring uses all four labels in a complementary pair on vertices in the same partite set; **(2)** No good coloring uses all three labels in a disjoint triple on vertices in the same partite set; **(3)** For any subset L of $\binom{[6]}{2}$ with $|L| = 7$, either L contains a complementary pair or it contains a disjoint triple. The theorem immediately follows from these claims by supposing G has a good coloring and letting L be the set of labels used on A.

(1) Suppose instead that the claim is false. By symmetry, labels $12, 34, 13$, and 24 all appear on vertices in A. The common neighbor of the vertices labeled 12 and 34 must receive label 56, as must the common neighbor of the vertices labeled 13 and 24. Since G is 3-regular, the two vertices labeled 56 are distinct; since they lie at distance 2, the coloring is invalid.

(2) Suppose instead that the claim is false. By symmetry, labels $12, 34$, and 56 all appear on vertices in A. These vertices cannot all have a common neighbor u, since then u would have no valid label. Thus they lie on a 6-cycle, and the three vertices of this 6-cycle in B must also have labels $12, 34$, and 56.

Consider a vertex $v \in A$ not adjacent to any vertex of this 6-cycle. (There is exactly one such vertex.) The label on v cannot be $12, 34$, or 56, so without loss of generality, it is 13. The common neighbor of v and the vertex in A having label 56 must have label 24, and the common neighbor of this vertex and the vertex in B having label 56 must have label 13. So two vertices in A have label 13; they must be distinct, since only one is adjacent to a vertex on the 6-cycle. Since they lie at distance 2, the coloring is invalid.

(3) Consider a color appearing in the most elements of L; without loss of generality, this color is 1. Let L_1 be the set of labels in L that contain 1. Note that $3 \leq |L_1| \leq 5$. We consider 3 cases.

If $|L_1| = 5$, then exactly two labels in L do not appear in L_1. If these labels are disjoint, then L contains a disjoint triple; otherwise, L contains a complementary pair.

If $|L_1| = 4$, then without loss of generality $L_1 = \{12, 13, 14, 15\}$. If two labels in L contain 6, then L contains a complementary pair. Similarly, if $L - L_1$ contains two non-disjoint labels not using 6, then L contains a complementary pair. Thus we may suppose that $L - L_1$ contains two disjoint labels not using 6 and one label using 6. Now the label using 6 is disjoint from one of the labels not using 6; these two labels, together with some label from L_1, form a disjoint triple.

If $|L_1| = 3$, then without loss of generality $L_1 = \{12, 13, 14\}$. Let $S_1 = \{23, 24, 34\}$, let $S_2 = \{25, 35, 45\}$, and let $S_3 = \{26, 36, 46\}$. If L contains two or more labels from any single S_i, then these labels, together with two labels from L_1, form a complementary pair. Thus we may suppose L contains exactly one label from each S_i and also contains the label 56. Now the label in $L \cap S_1$, the label 56, and some element of L_1 form a disjoint triple. ∎

Below we give a 2-tone 7-coloring of the Heawood graph, which completes the proof that its 2-tone chromatic number is 7.

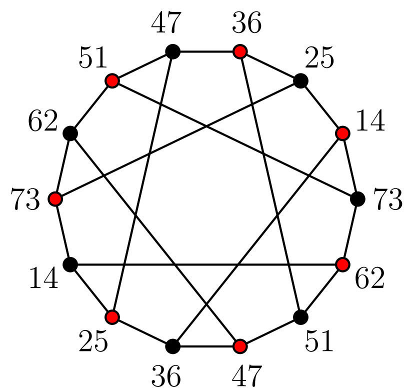

Fig. 1: A 2-tone 7-coloring of the Heawood graph.

We next show that $\tau_2(G) \leq 8$ whenever $\Delta(G) \leq 3$, thus verifying part (a) of Conjecture 1.3. The proof requires careful attention to detail, so we isolate some of the more delicate arguments in lemmas. Before stating the lemmas, we introduce some terminology.

Definition 2.7 *Let f be a partial 2-tone coloring of a graph G and let v be an uncolored vertex. A* valid *label for v is a label by which f can be extended to v. A* free *color at v is one not appearing on any neighbor of v. A* candidate *label for v is a label containing only free colors. An* obstruction *of v is a candidate label that is not valid (because it appears on some second-neighbor of G).*

Our first lemma is short and simple, but provides a good introduction to the techniques that appear throughout the proof.

Lemma 2.8 *Let G be a graph with maximum degree at most 3. Let f be a partial 2-tone 8-coloring of G and let v be an uncolored vertex. If v has at least one uncolored neighbor and at least one uncolored second-neighbor, then f can be extended to v.*

Proof. At least four colors are free at v, so it has at least six candidate labels. Since v has an uncolored second-neighbor, v has at most five obstructions, so some candidate is valid. ∎

In the main proof we first color all vertices except for those on some induced cycle C; we then iteratively extend our partial coloring along C. We will need to maintain some flexibility while doing so, and the next two lemmas provide this desired freedom.

Lemma 2.9 *Let G be a 3-regular graph, let v be a vertex of G, and let w_1 and w_2 be distinct neighbors of v. Let f be a partial 2-tone 8-coloring of G that leaves v, w_1, and w_2 uncolored, and let f_1 and f_2 be distinct extensions of f to w_1. If two second-neighbors of v do not yield obstructions under any f_i, then some f_i can be extended to v in three different ways.*

Proof. Let S_i be the set of free colors at v under f_i. Under each f_i, at most four colors appear on neighbors of v, so $|S_i| \geq 4$. Either some S_i contains at least five colors, or $S_1 \neq S_2$; in either case, the f_i yield at least nine candidate labels between them. Since v has at most four obstructions, the two f_i together yield at least five valid labels, so by the Pigeonhole Principle some f_i admits three extensions to v. ■

Lemma 2.10 *Let G be a 3-regular graph, let v be a vertex of G, and let w_1 and w_2 be distinct neighbors of v. Let f be a partial 2-tone 8-coloring of G that leaves v, w_1, and w_2 uncolored, and let f_1, f_2, and f_3 be distinct extensions of f to w_1. If some second-neighbor of v does not yield an obstruction under any f_i, then some f_i can be extended to v in three different ways.*

Proof. Let S_i be the set of free colors at v under f_i. Under each f_i, at most four colors appear on neighbors of v, so $|S_i| \geq 4$. If some S_i contains five or more colors, then v has at least ten candidate labels and at most five obstructions under f_i, so f_i admits at least five extensions to v. Otherwise, since the f_i assign different labels to w_1, no two S_i are the same. Since v has at least six candidate labels under each f_i, it suffices to show that v cannot have four obstructions under each f_i simultaneously.

Without loss of generality, $S_1 = \{1, 2, 3, 4\}$. Since $S_2 \neq S_1$, we may assume $5 \in S_2$. If additionally S_2 contains some other color not in S_1, then at most one label is a candidate under both f_1 and f_2; in this case v has at most one common obstruction under f_1 and f_2, so it cannot have four obstructions under both f_1 and f_2. Hence we may assume $S_2 = \{1, 2, 3, 5\}$. Now f_1 and f_2 yield three common candidates, namely $12, 13$, and 23; if v does not have three valid labels under either f_i, then all three common candidates must be obstructions. Moreover, of the two remaining obstructions, one lies in $\{14, 24, 34\}$ and the other in $\{15, 25, 35\}$. If S_3 contains $1, 2$, and 3, then without loss of generality $S_3 = \{1, 2, 3, 6\}$, and f_3 can be extended via 16, 26, and 36. Otherwise at most one of 12, 13, and 23 is an obstruction under f_3, and again f_3 admits three extensions to v. ■

Our final lemma helps us leverage the flexibility ensured by Lemma 2.10 to complete a partial coloring.

Lemma 2.11 *Let G be a 3-regular graph. Let v be a vertex of G, let w_1, w_2, and w_3 be its neighbors, and let x be one of its second-neighbors. Let f be a partial 2-tone 8-coloring of G that leaves v and w_1 uncolored, and under which w_2 shares one color with w_3 and one with x. If f has three extensions to w_1, then one of these extensions can itself be extended to v.*

Proof. Let f_1, f_2, and f_3 be extensions of f to w_1. Since w_2 and x share a color, x cannot yield an obstruction of v, so v has at most five different obstructions between all three f_i. Since w_2 and w_3 share a color, at most five colors appear on neighbors of v in each f_i, hence always at least three colors are free at v. Let S_i be the set of free colors at v under f_i. If any S_i contains at least four colors, then v has at least six candidate labels under f_i, one of which must be valid. Otherwise, each S_i has size three; moreover, since the f_i differ in the colors they assign to w_1, no two S_i are identical. S_1 and S_2 together yield at least five different candidate labels for v, and S_3 yields a sixth; again we have six candidate labels, one of which must be valid. Thus some f_i can be extended to v. ∎

We are now ready to present the main proof.

Theorem 2.12 *If G is a graph with $\Delta(G) \leq 3$, then $\tau_2(G) \leq 8$.*

Proof. Suppose otherwise, and let G be a smallest counterexample. Clearly G is connected and is not K_4.

Suppose that G is not 3-regular, and let v be a vertex of degree 1 or 2. By Lemma 2.8, iteratively coloring in non-increasing order of distance from v yields a partial 2-tone 8-coloring of G leaving only $N[v]$ uncolored. Each neighbor u of v now has at least four free colors (hence at least six candidate labels) and at most five second-neighbors, so we may extend the coloring to u. Likewise, v itself now has at least four free colors and at most four second-neighbors, so we may extend to v as well, completing the coloring and contradicting the choice of G. Hence G must be 3-regular.

Next suppose that G contains an induced $K_{2,3}$. Let x_1, x_2, y_1, y_2, and y_3 be the vertices of this $K_{2,3}$, with the x_i the vertices of degree 3 and the y_i the vertices of degree 2; let u_i be the third neighbor of each y_i. Let $G' = G - \{x_1, x_2, y_1, y_2, y_3\}$. Since G' is not 3-regular, it has a 2-tone 8-coloring, which is also a partial 2-tone 8-coloring of G. Without loss of generality, the color 1 does not appear on any u_i. We aim to color each y_i with a label containing color 1; each y_i has five such candidate labels and at most four second-neighbors, so this is possible. Now each x_i has at least four free colors, and hence at least six candidate labels. Since each x_i has at most four second-neighbors, we may extend the coloring to each x_i in turn, again contradicting the choice of G. Thus G is $K_{2,3}$-free.

Let C be a shortest cycle in G; label its vertices $v_1, \ldots, v_k$ in cycle order. Let $u_1, \ldots, u_k$ be the neighbors off C of $v_1, \ldots, v_k$, respectively. The u_i need not be distinct, but (since $G \neq K_4$) cannot all be the same vertex. If C is a triangle, then without loss of generality $u_1 \neq u_2$. If not, then for all i we have $u_{i-1} \neq u_{i+1}$: if C is a four-cycle then this follows from the fact that G is $K_{2,3}$-free, and otherwise it follows from the minimality of C. In any case, construct G' from G by deleting the vertices of C and adding the edge $u_{k-1}u_1$ (if it is not already present); if C is not a triangle, then add the edge $u_k u_2$ as well. By the minimality of G, the graph G' is 2-tone 8-colorable. A 2-tone 8-coloring of G' is also a partial 2-tone 8-coloring of G in which only the v_i are uncolored and in which u_{k-1} and u_1 have disjoint labels; if C has at least four vertices, then also u_k and u_2 have disjoint labels. We use such a coloring as a starting point in producing a 2-tone 8-coloring of G.

We have three cases to consider. **(1)** If the label on u_k is identical to one of the labels on u_{k-1} or u_1, then by symmetry we may suppose that u_{k-1}, u_k, and u_1 have labels $12, 12$, and 34. **(2)** If the label on u_k is disjoint from the labels on u_{k-1} and u_1, then we may suppose that u_{k-1}, u_k, and u_1 have labels $12, 34$, and 56. **(3)** Otherwise, we may suppose that u_{k-1}, u_k, and u_1 have labels $12, 13$, and L, where $1 \notin L$.

Case (1): *u_{k-1}, u_k, and u_1 have labels 12, 12, 34.* We aim to assign v_1 a label containing either 1 or 2; v_1 has nine such candidate labels, and it has at most four obstructions, so at least five such labels are valid. Since we have at least three ways to extend to v_1, by Lemma 2.10, we subsequently have at least three ways to extend to v_2, then to v_3, and so on up to v_{k-2}. Since the labels on u_{k-1} and v_1 have nonempty intersection, v_1 cannot yield an obstruction of v_{k-1}, so again we have three ways to extend to v_{k-1}. Now applying Lemma 2.11 (with $v = v_k, w_1 = v_{k-1}, w_2 = v_1, w_3 = u_k$, and $x = u_{k-1}$) lets us complete the coloring.

Case (2): *u_{k-1}, u_k, and u_1 have labels 12, 34, 56.* First suppose that C is a triangle. Give v_1 a label from $\{13, 23, 37, 38\}$; since v_1 has at most two obstructions, this is possible. Next give v_2 a label from $\{45, 46, 47, 48\}$; at most one of these labels has nonempty intersection with the label on v_1, and v_2 has at most two additional obstructions, so again some such label is valid. We have ensured that four colors remain free at v_3. Thus v_3 has six candidate labels and at most four obstructions, so we can complete the coloring.

Suppose now that C is not a triangle. We aim to assign v_1 a label from $\{13, 14, 23, 24\}$. Although v_1 has four colored second-neighbors, u_k has label 34, which is not an obstruction. Moreover, by construction the label on u_2 contains neither 3 nor 4, so it also cannot be an obstruction. Thus, at least two such labels are valid. By Lemma 2.9, this coloring admits three extensions to v_2. Now we may apply Lemma 2.10 and Lemma 2.11 (with $v = v_k, w_1 = v_{k-1}, w_2 = v_1, w_3 = u_k$, and $x = u_{k-1}$) as before to complete the coloring.

Case (3): *u_{k-1}, u_k, and u_1 have labels $12, 13, L$, where $1 \notin L$.* We aim to give v_1 a label containing either 1 or 3. If $3 \notin L$, then v_1 has at least nine such candidates and at most four obstructions, so at least five of the candidates are valid. Otherwise v_1 has only five such candidate labels, but u_k does not yield an obstruction, so at least two of these candidates are valid. In each case, by Lemma 2.9 we may extend the coloring to v_2 in at least three different ways. Now by Lemma 2.10 and Lemma 2.11 (with $v = v_k, w_1 = v_{k-1}, w_2 = u_k, w_3 = v_1$, and $x = u_{k-1}$) we can again complete the coloring. ∎

3 General t-tone Coloring

We next study the behavior of τ_t for general t. We have already noted that $\tau_t(G)$ is monotone in G; that is, $\tau_t(H) \leq \tau_t(G)$ whenever H is a subgraph of G. It is also true that $\tau_t(G)$ is monotone in t.

Proposition 3.1 *If $t < t'$ and G is any graph, then $\tau_t(G) \leq \tau_{t'}(G)$.*

Proof. Given a graph G and a t'-tone coloring of G, we arbitrarily discard $t' - t$ colors from each label of G. This yields a t-tone coloring, since the process cannot increase the size of the intersection of any two labels. ∎

Our first main result in this section is a generalization of Theorem 2.2. In the case $t = 2$, Theorem 2.2 gives a better bound, since restricting to $t = 2$ allows tighter analysis.

Theorem 3.2 *For every integer t and every nonempty graph G, we have $\tau_t(G) \le (t^2 + t)\Delta(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $r = \Delta(G)$, and let $k = (t^2 + t)r$. As in the proof of Theorem 2.2, we construct a t-tone k-coloring of G by coloring iteratively with respect to the ordering $v_1, \ldots, v_n$.

When coloring v_i, at most tr colors appear on neighbors of v_i, so at least t^2r other colors remain. We have $\binom{t^2r}{t}$ labels that use only these colors, and each is a candidate label for v_i.

Given a label L, we say that vertex u *forbids* L if L and the label on u have intersection size at least $d(u, v_i)$. Recall that we have already discarded all labels forbidden by neighbors of v_i. For $2 \le d \le t$, each vertex at distance d from v_i forbids at most $\binom{t}{d}\binom{t^2r-d}{t-d}$ labels. At most $r(r-1)^{d-1}$ vertices lie at distance d from v_i, so to show that we may color v_i, it suffices to show that

$$\sum_{d=2}^{t} \binom{t}{d}\binom{t^2r-d}{t-d} r(r-1)^{d-1} < \binom{t^2r}{t},$$

or equivalently, that

$$\sum_{d=2}^{t} \frac{\binom{t}{d}\binom{t^2r-d}{t-d} r(r-1)^{d-1}}{\binom{t^2r}{t}} < 1.$$

Ultimately, we will show that the dth term of the sum is less than $1/d!$, and thus (since $1/d! \le 2^{1-d}$) the sum is less than 1. We first simplify each term. For fixed d,

$$\begin{aligned}
\frac{\binom{t}{d}\binom{t^2r-d}{t-d} r(r-1)^{d-1}}{\binom{t^2r}{t}} &= \frac{t!}{d!(t-d)!} \cdot \frac{(t^2r-d)!}{(t-d)!(t^2r-t)!} \cdot r(r-1)^{d-1} \cdot \frac{t!(t^2r-t)!}{(t^2r)!} \\
&= \frac{1}{d!} \cdot \left(\frac{t!}{(t-d)!}\right)^2 \cdot \frac{(t^2r-d)!}{(t^2r)!} \cdot r(r-1)^{d-1} \\
&= \frac{1}{d!} \frac{\left(t(t-1)(t-2)\cdots(t-d+1)\right)^2 r(r-1)^{d-1}}{t^2r(t^2r-1)\cdots(t^2r-d+1)} \\
&= \frac{1}{d!} \cdot \frac{(t-1)^2(r-1)}{t^2r-1} \cdot \frac{(t-2)^2(r-1)}{t^2r-2} \cdots \frac{(t-d+1)^2(r-1)}{t^2r-d+1}.
\end{aligned}$$

Now for i between 1 and $d-1$, we have

$$(t-i)^2(r-1) < (t-i)^2r = t^2r - i(2t-i)r \le t^2r - i,$$

hence

$$\frac{\binom{t}{d}\binom{t^2r-d}{t-d} r(r-1)^{d-1}}{\binom{t^2r}{t}} < \frac{1}{d!} \cdot 1 \cdot 1 \cdots 1 = \frac{1}{d!}.$$

Now

$$\sum_{d=2}^{t} \frac{\binom{t}{d}\binom{t^2r-d}{t-d} r(r-1)^{d-1}}{\binom{t^2r}{t}} < \sum_{d=2}^{t} \frac{1}{d!} \leq \sum_{d=2}^{t} \frac{1}{2^{d-1}} < 1,$$

which completes the proof. ■

In [4] it was shown that for every tree T, we have $\tau_2(T) = \left\lceil (5 + \sqrt{1 + 8\Delta(T)})/2 \right\rceil$. By Proposition 3.1, it thus follows that $\tau_t(T) \geq \left\lceil (5 + \sqrt{1 + 8\Delta(T)})/2 \right\rceil$ whenever $t \geq 2$. In fact this bound is asymptotically best possible, as we show next.

Theorem 3.3 *For every positive integer t, there exists a constant $c = c(t)$ such that for every nontrivial tree T we have $\tau_t(T) \leq c\sqrt{\Delta(T)}$.*

Proof. Fix a positive integer t and a tree T. Let $k = \sqrt{\Delta(T)}$. Let T' be the complete $(\Delta(T) - 1)$-ary tree of height $|V(T)|$; that is, T' is a rooted tree such that all vertices at distance $|V(T)|$ from the root are leaves, and all others have $\Delta(T) - 1$ children. By *level i* of T' we mean the set of vertices at distance i from the root. Clearly T is contained in T', so by monotonicity of τ_t it suffices to prove that $\tau_t(T') \leq ck$ for some constant c (to be defined later, but independent of T). Moreover, by Proposition 3.1, we may assume that t is even.

A *palette* is a set of colors. We color T' using $t + 1$ disjoint palettes, each of size at most c_1k for some constant c_1. On level i of the tree we use only those colors in the ith palette (with i taken modulo $t + 1$). This restriction ensures that whenever u and v are within distance t of each other, either they lie on the same level of T' or they receive colors from different palettes (and hence have disjoint labels). Thus, we need only consider a single level of T' and show that the vertices on that level can be colored using at most c_1k colors.

Within each level, color iteratively with respect to an arbitrary vertex ordering. Note that any two vertices on the same level of T' lie at an even distance. Fix a vertex v and an integer d between 1 and $t/2$. Given a label L, say that vertex u *forbids* L if L and the label on u have intersection size at least $d(u, v)$. The number of vertices at distance $2d$ from v, and on the same level as v, is bounded above by $[\Delta(T)]^d$ and hence by k^{2d}; each such vertex forbids at most $\binom{t}{2d}\binom{c_1k-2d}{t-2d}$ labels in $\binom{[c_1k]}{t}$. Thus the total number of forbidden labels is at most

$$\sum_{d=1}^{t/2} k^{2d} \binom{t}{2d} \binom{c_1k - 2d}{t - 2d},$$

which is at most

$$k^t \sum_{d=1}^{t/2} \frac{t^{2d} c_1^{t-2d}}{(2d)!(t - 2d)!}.$$

We have $\binom{c_1k}{t}$ available labels; for fixed t and large k, this is at least $k^t \frac{(c_1-1)^t}{t!}$. For sufficiently large c_1 we have

$$\frac{(c_1 - 1)^t}{t!} > \sum_{d=1}^{t/2} \frac{t^{2d} c_1^{t-2d}}{(2d)!(t - 2d)!},$$

since both sides of the inequality are polynomials in c_1, but the left side has higher degree. Thus if c_1 is large enough, then we can color v. ■

A graph is *k-degenerate* if each of its subgraphs contains a vertex of degree at most k; trees are precisely the connected 1-degenerate graphs. For $k \geq 2$, on the class of k-degenerate graphs we can improve the bound given by Theorem 3.2.

Lemma 3.4 *If G is a k-degenerate graph, then G has a vertex ordering such that, for each integer $d \geq 1$ and for each vertex v, at most $dk\Delta(G)(\Delta(G)-1)^{d-2}$ vertices preceding v in the ordering lie at distance d from v.*

Proof. Construct an ordering of $V(G)$ by repeatedly deleting a vertex v of minimum degree and prepending v to the ordering. We claim that this ordering has the desired properties.

Fix v and consider the set of earlier vertices at distance d from v. Each such vertex can be reached from v via a walk of length d in which at least one step moves backward in the ordering. For each i between 1 and d, there are at most $k\Delta(G)(\Delta(G)-1)^{d-2}$ such walks that move backward on step i, since we have at most k choices for the ith step, at most $\Delta(G)$ choices for the first, and at most $\Delta(G)-1$ choices for each of the others. ■

When d is large, the bound in Lemma 3.4 is worse than the easy bound of $\Delta(G)(\Delta(G)-1)^{d-1}$ that holds for all graphs G, regardless of degeneracy. However, when applying Lemma 3.4, we will mainly care about small values of d.

Theorem 3.5 *If G is a k-degenerate graph, $k \geq 2$, and $\Delta(G) \leq r$, then for every t we have $\tau_t(G) \leq kt + kt^2 r^{1-1/t}$.*

Proof. Let $c = kt^2 r^{1-1/t}$. Let $v_1, \ldots, v_n$ be a vertex ordering of the form guaranteed by Lemma 3.4; we construct a t-tone $(c+kt)$-coloring of G by coloring iteratively with respect to this ordering.

When coloring v_i, as many as kt colors may appear on v_i's neighbors; at least c other colors remain. Thus v_i has at least $\binom{c}{t}$ candidate labels using these c colors. As in the proof of Theorem 3.2, say that a vertex u *forbids* a label L if L and the label on u have intersection of size at least $d(u, v_i)$. By Lemma 3.4, at most $dkr(r-1)^{d-2}$ colored vertices lie at distance d from v_i; each such vertex forbids at most $\binom{t}{d}\binom{c-d}{t-d}$ of the candidates. Thus to show that we can color v_i, it suffices to show that

$$\sum_{d=2}^{t} \binom{t}{d}\binom{c-d}{t-d} dkr(r-1)^{d-2} < \binom{c}{t},$$

or equivalently, that

$$\sum_{d=2}^{t} \frac{\binom{t}{d}\binom{c-d}{t-d} dkr(r-1)^{d-2}}{\binom{c}{t}} < 1.$$

We proceed as in the proof of Theorem 3.2.

$$\begin{aligned}
\frac{\binom{t}{d}\binom{c-d}{t-d}dkr(r-1)^{d-2}}{\binom{c}{t}} &= \frac{t!}{d!(t-d)!}\cdot\frac{(c-d)!}{(t-d)!(c-t)!}\cdot dkr(r-1)^{d-2}\cdot\frac{t!(c-t)!}{c!} \\
&= \frac{dk}{d!}\cdot\left(\frac{t!}{(t-d)!}\right)^2\cdot\frac{(c-d)!}{c!}\cdot r(r-1)^{d-2} \\
&< \frac{k}{(d-1)!}\cdot\frac{(t(t-1)\cdots(t-d+1))^2}{c(c-1)\cdots(c-d+1)}\cdot r^{d-1} \\
&= \frac{k}{(d-1)!}\cdot\frac{t^2r^{1-1/d}}{kt^2r^{1-1/t}}\cdots\frac{(t-d+1)^2r^{1-1/d}}{kt^2r^{1-1/t}-d+1} \\
&\le \frac{1}{(d-1)!k^{d-1}}\cdot\frac{t^2r^{1-1/d}}{t^2r^{1-1/t}}\cdots\frac{(t-d+1)^2r^{1-1/d}}{t^2r^{1-1/t}-d+1}
\end{aligned}$$

For s between 0 and $d-1$, we have

$$(t-s)^2r^{1-1/d} \le (t-s)^2r^{1-1/t} = t^2r^{1-1/t} - s(2t-s)r^{1-1/t} \le t^2r^{1-1/t} - s,$$

so

$$\frac{\binom{t}{d}\binom{c-d}{t-d}dkr(r-1)^{d-2}}{\binom{c}{t}} < \frac{1}{(d-1)!k^{d-1}}.$$

Thus

$$\sum_{d=2}^{t}\frac{\binom{t}{d}\binom{c-d}{t-d}dkr(r-1)^{d-2}}{\binom{c}{t}} < \sum_{d=2}^{t}\frac{1}{(d-1)!k^{d-1}} < 1,$$

as desired. ■

Fonger, Goss, Phillips, and Segroves [4] showed that $\tau_2(K_{1,k}) = \Theta(\sqrt{k})$. Thus by Proposition 3.1, the bound in Theorem 3.5 is asymptotically tight (in terms of $\Delta(G)$) when $t = 2$.

We have made several statements about the asymptotics of $\tau_t(G)$ when t is fixed and $\Delta(G)$ grows; we now consider what happens when $\Delta(G)$ is fixed and t grows. The bound in Theorem 3.2 shows that, for fixed values of $\Delta(G)$, we have $\tau_t(G) \le ct^2$ for some constant c. Our final result shows that the asymptotics of this bound cannot be improved much, if at all.

Theorem 3.6 *For each $r \ge 3$, there exists a constant c such that for all t, there is a graph G for which $\Delta(G) = r$ and $\tau_t(G) \ge ct^2/\lg t$.*

Proof. Let G be the complete $(r-1)$-ary tree of height $\lceil \lg t \rceil$. Consider a t-tone coloring of G; examine the vertices of G in any order. Since any two vertices of G lie within distance $2\lceil \lg t \rceil$, each vertex we examine shares fewer than $2\lceil \lg t \rceil$ colors with each vertex already examined. Thus, the number of colors used in this coloring is at least

$$\sum_{i=0}^{|V(G)|-1}\max\{0, t - 2\lceil \lg t \rceil\, i\}.$$

When $i \leq t/(4\lceil \lg t \rceil)$, the ith term of this sum is at least $t/2$. Note that $|V(G)| > (r-1)^{\lg t} \geq t > t/(4\lceil \lg t \rceil)$, so the sum has at least $t/(4\lceil \lg t \rceil)$ terms. Thus, the number of colors used is at least $t^2/(8\lceil \lg t \rceil)$. ∎

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