Multicolor on-line degree Ramsey numbers of trees

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Abstract

In the on-line Ramsey game on a family $\mathcal{H}$ of graphs, "Builder" presents edges of a graph one-by-one, and "Painter" colors each edge as it is presented; we require that Builder keep the presented graph in $\mathcal{H}$. Builder wins the game $(G, \mathcal{H})$ if Builder can ensure that a monochromatic G arises. The *s-color on-line degree Ramsey number of* G, denoted $\mathring{R}_\Delta(G; s)$, is the least k such that Builder wins $(G, \mathcal{H})$ when $\mathcal{H}$ is the family of graphs having maximum degree at most k and Painter has s colors available. More generally, $\mathring{R}_\Delta(G_1, \ldots, G_s)$ is the minimum k such that Builder can force a copy of G_i in color i for some i when restricted to graphs with maximum degree at most k.

In this paper, we prove that $\mathring{R}_\Delta(T; s) \le s(\Delta(T) - 1) + 1$ for every tree T; this is sharp, with equality whenever T has adjacent vertices of maximum degree. We also give lower and upper bounds on $\mathring{R}_\Delta(G_1, \ldots, G_s)$ when each G_i is a double-star. When each G_i is a star, we determine $\mathring{R}_\Delta(G_1, \ldots, G_s)$ exactly.

1 Introduction

When every 2-edge-coloring of a host graph H contains a monochromatic copy of a target graph G, we write $H \to G$. More generally, when every s-edge-coloring of H contains a monochromatic G, we write $H \xrightarrow{s} G$. The central problem of graph Ramsey theory is to find the least n such that $K_n \to G$, which can be restated as $\min\{|V(H)|: H \to G\}$. The value is called the *Ramsey number* of G, denoted $R(G)$ (or $R(G; s)$ in the s-color setting).

This phrasing of the Ramsey number generalizes: given a graph parameter ρ, let $R_\rho(G) = \min\{\rho(H): H \to G\}$. When ρ is the maximum degree, the Ramsey parameter is the *degree Ramsey number*, $R_\Delta(G)$. Burr, Erdős, and Lovász [2] introduced this notion and determined $R_\Delta(K_{1,k})$ and $R_\Delta(K_n)$; further results appear in [7, 9].

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An on-line variant of the degree Ramsey number can be phrased as a game played by two players, "Builder" and "Painter", on an infinite set of vertices. In each round, Builder introduces an edge and Painter colors it from a fixed set of s colors. Builder aims to force a monochromatic copy of a target graph G. By Ramsey's Theorem, Builder can win by presenting a large complete graph. Thus we restrict Builder by requiring that the presented graph remains in a family $\mathcal{H}$; the game is then *played on* $\mathcal{H}$. If Builder can still force a monochromatic G, then we say Builder wins. More generally, Builder *wins* $(G_1, \ldots, G_s; \mathcal{H})$ if Builder can force a copy of G_i in color i for some i when playing on $\mathcal{H}$ with Painter having s colors available. Introduced by Beck [1], this model was studied by Grytczuk, Hałuszczak, and Kierstead [5] for several natural choices of $\mathcal{H}$ in the case where $s = 2$ and $G_1 = G_2$. Later results appear in [4, 6, 8, 10].

We focus on the case where $\mathcal{H}$ is $\mathcal{S}_k$, the set of graphs with maximum degree at most k. We define $\mathring{R}_\Delta(G_1, \ldots, G_s)$ to be the least k such that Builder wins $(G_1, \ldots, G_s; \mathcal{S}_k)$. When $G_1 = \cdots = G_s = G$, we have the *diagonal* case and abbreviate the notation to $\mathring{R}_\Delta(G; s)$, called the *$s$-color on-line degree Ramsey number* of G. The parameter is well-defined, since it is bounded by the ordinary s-color Ramsey number minus 1.

For $s = 2$, Butterfield et al. [3] determined the exact 2-color on-line degree Ramsey numbers for paths, stars, and double-stars (trees with diameter 3), and they proved that $\mathring{R}_\Delta(T; 2) \leq 2\Delta(T) - 1$ for every tree T. In this paper, we extend several of those results to the s-color, non-diagonal setting. Proposition 2.3 states that $\mathring{R}_\Delta(P_{n_1}, \ldots, P_{n_s}) = s + 1$ when each n_i is at least 4; this uses a recursive lower bound for $\mathring{R}_\Delta(G_1, \ldots, G_s)$ in terms of $\mathring{R}_\Delta(G_1, \ldots, G_{s-1})$. Theorem 2.5 gives somewhat technical lower and upper bounds for $\mathring{R}_\Delta(G_1, \ldots, G_s)$ when each G_s is a double-star; these bounds coincide when the central vertices of each G_i have identical degrees (Corollary 2.7). A refined argument determines the exact value when each G_i is a star (Theorem 2.8). Finally, Theorem 2.10 states that $\mathring{R}_\Delta(T_1, \ldots, T_s) \leq \sum_{i=1}^{s}(\Delta(T_i) - 1) + 1$ when each T_i is a tree; this bound holds with equality when each T_i has adjacent vertices of maximum degree.

2 Results

In the course of a particular game, we often focus attention on special subgraphs of the presented graph, usually monochromatic. In such situations, we must distinguish between the degree of a vertex within the subgraph and its degree within the full presented graph. We use "degree" to mean "degree within the given subgraph" and "global degree" to mean "degree within the full presented graph".

In giving strategies for Builder to prove upper bounds, we may assume that Painter behaves "consistently". A *consistent Painter* chooses a color for edge uv based solely on the edge-colored components presently containing u and v. It was shown in [3] that for any graph

G and any monotone additive graph family $\mathcal{H}$, Builder wins $(G;\mathcal{H})$ if and only if Builder wins against every consistent Painter. Thus consistent Painters are no weaker than general Painters, but this formal restriction on the Painter simplifies what needs to be said for a Builder strategy. If Builder repeats the same sequence of moves on disjoint sets of vertices, then a consistent Painter produces the same coloring every time. This observation yields the lemma below, which we apply throughout the paper without explicit citation.

Lemma 2.1. *If Builder can force an edge-colored graph G against a consistent Painter, then Builder can force arbitrarily many copies of G.* □

Our first result is a general lower bound on $\mathring{R}_\Delta(G;s)$. It uses a Painter strategy that generalizes the "greedy $\mathcal{S}_k$-Painter" from [3], who colors an edge red when the resulting red subgraph would belong to $\mathcal{S}_k$ and blue otherwise.

Proposition 2.2. *For graphs $G_1,\dots,G_s$,*

$$\mathring{R}_\Delta(G_1,\dots,G_s) \geq (\mathring{R}_\Delta(G_1,\dots,G_{s-1})-1) + \max_{uv\in E(G_s)} \min\{d_{G_s}(u), d_{G_s}(v)\}.$$

Proof. Let $d=\mathring{R}_\Delta(G_1,\dots,G_{s-1})-1$ and $r=\max_{uv\in E(G_s)}\min\{d_{G_s}(u),d_{G_s}(v)\}$; we provide a strategy for Painter to win on $\mathcal{S}_{d+r-1}$. Painter colors edges using blue and $s-1$ shades of red. Painter behaves similarly to a greedy $\mathcal{S}_d$-Painter. However, whenever Painter colors an edge red, he chooses the particular shade of red according to some winning strategy for $(G_1,\dots,G_{s-1};\mathcal{S}_d)$. In this way Painter avoids producing a copy of any G_i in the corresponding shade of red; it suffices to show that also Painter produces no blue G_s.

Suppose that Painter has produced a blue copy H of G_s. Choose an edge uv in H maximizing $\min\{d_H(u),d_H(v)\}$. Since Painter colored uv blue, one of its endpoints, say u, lies on d red edges in the presented graph. Since u also lies on at least $d_H(u)$ blue edges, it has global degree at least $d+r$, a contradiction. □

As an application of Proposition 2.2, we determine $\mathring{R}_\Delta(P_{n_1},\dots,P_{n_s})$; the proof of this result introduces techniques used in the proof of Theorem 2.10.

Proposition 2.3. $\mathring{R}_\Delta(P_{n_1},\dots,P_{n_s})=s+1$ *when $n_i\geq 4$ for all i in $\{1,\dots,s\}$.*

Proof. Letting $n=\max\{n_1,\dots,n_s\}$, it suffices to prove that $\mathring{R}_\Delta(P_n;s)=s+1$ when $n\geq 4$. The lower bound follows from Proposition 2.2 and the observation that $\mathring{R}_\Delta(P_n;1)=2$.

For the upper bound, we provide a strategy for Builder. We use induction on s; the observation above establishes the case $s=1$. Suppose that Builder can force P_n on $\mathcal{S}_s$ when Painter has $s-1$ colors available. Consider an s-color game on $\mathcal{S}_{s+1}$. A consistent Painter uses the same color on all isolated edges; without loss of generality, call this color blue. Let the other $s-1$ colors be shades of red. It suffices to show that for any k, Builder can force either a P_n in some shade of red or a blue P_{2^k} in which each endpoint has global degree 1.

We prove this claim by induction on k. The case $k = 1$ is immediate, since Painter colors isolated edges blue. For the induction step, Builder first forces many blue copies of $P_{2^{k-1}}$ whose endpoints have global degree 1. Builder next selects one endpoint from each of these blue paths. On these endpoints, Builder plays a winning strategy for the $(s-1)$-color game $(P_n; \mathcal{S}_s)$, provided by the overall induction hypothesis (the global degree remains at most $s+1$). If Painter uses only the $s-1$ shades of red, then P_n arises in some shade of red. Otherwise, Painter colors some edge blue; this connects two blue paths, yielding a blue P_{2^k} whose endpoints still have global degree 1. □

We next consider stars and double-stars.

Definition 2.4. A *double-star* is a tree with diameter 3. Such a tree has two central vertices; we denote by $S_{a,b}$ the double-star with central vertices of degrees a and b.

Theorem 2.5. *If $a_i \le b_i$ for all i in $\{1, \ldots, s\}$, then*

$$b_1 - 1 + \sum_{i=2}^{s}(a_i - 1) + 1 \le \mathring{R}_\Delta(S_{a_1,b_1}, \ldots, S_{a_s,b_s}) \le \min_{X \subseteq \{1,\ldots,s\}} f_X(a_1, \ldots, a_s, b_1, \ldots, b_s),$$

where

$$f_X(a_1, \ldots, a_s, b_1, \ldots, b_s) = 1 + \max\left\{\sum_{i\in X}(b_i - 1) + \sum_{j\notin X}(a_j - 1), \sum_{i\in X}(a_i - 1) + \sum_{j\notin X}(b_j - 1)\right\}.$$

Proof. The lower bound follows by induction on s, using Proposition 2.2 and the observation that $\mathring{R}_\Delta(S_{a,b}; 1) = b$ when $a \le b$.

To establish the upper bound, we provide a strategy for Builder. Builder first partitions the set of available colors into some sets X and Y. Builder aims to make two special vertices u and v the central vertices of a monochromatic double-star. Let the *quota* of u in color i be $b_i - 1$ if $i \in X$ and $a_i - 1$ if $i \in Y$. For v, use the reverse values: the quota of v in color i is $a_i - 1$ if $i \in X$ and $b_i - 1$ if $i \in Y$. Whenever u or v reaches its quota of incident edges in a color c, call that vertex *saturated in color c*. Note that coloring uv with a color in which both u and v are saturated produces the desired monochromatic double-star in that color.

Starting with u and v as isolated vertices, Builder repeats the following process for the remainder of the game. Let G_u and G_v denote the current components of the presented graph that contain u and v, respectively. Builder presents edge uv; let c be the color Painter uses on it. If u was not already saturated in c, then Builder adds uv and all of G_v to G_u, creates new copies of v and G_v, and repeats. If u was saturated in c but v was not, then Builder adds uv and all of G_u to G_v, creates new copies of u and G_u, and repeats. Finally, if both u and v were already saturated in c, then u and v are now the central vertices of a monochromatic S_{a_c,b_c} in color c, and Builder has won.

Whenever G_u or G_v is enlarged, the special vertex receives another incident edge, so always u or v has maximum global degree after G_u and G_v are "recreated". When Builder is ready to present the edge uv, the degree of u is at most $\sum_{i\in X}(b_i-1)+\sum_{j\notin X}(a_j-1)$, and the degree of v is at most $\sum_{i\in X}(a_i-1)+\sum_{j\notin X}(b_j-1)$. Thus the maximum degree used is at most one more than the maximum of these two quantities. Optimizing over the choice of X yields the stated bound. □

In the diagonal case, the minimum over X in the upper bound in Theorem 2.5 occurs whenever $|X| = \lceil s/2 \rceil$. This yields a much simpler formula:

Corollary 2.6. *If $a \le b$, then $\mathring{R}_\Delta(S_{a,b}; s) \le \lceil s/2 \rceil (b-1) + \lfloor s/2 \rfloor (a-1) + 1$.* □

When $a_i = b_i$ for all i, the upper and lower bounds in Theorem 2.5 coincide:

Corollary 2.7. *For "symmmetric" double stars, $\mathring{R}_\Delta(S_{b_1,b_1}, \ldots, S_{b_s,b_s}) = \sum_{i=1}^{s}(b_i-1)+1$. In particular, $\mathring{R}_\Delta(S_{b,b}; s) = s(b-1)+1$.* □

When each double-star is in fact a star, the upper bound in Theorem 2.5 is the correct value. The answer is obtained by splitting the sum of the sizes of the stars as equally as possible and using the larger half in such a split.

Theorem 2.8. $\mathring{R}_\Delta(K_{1,k_1}, \ldots, K_{1,k_s}) = 1 + \min_{X\subseteq\{1,\ldots,s\}} \max\left\{\sum_{i\in X}(k_i-1), \sum_{i\notin X}(k_i-1)\right\}$. *In particular, $\mathring{R}_\Delta(K_{1,k}; s) = \left\lceil \frac{s}{2} \right\rceil (k-1)+1$.*

Proof. The upper bound follows from Theorem 2.5.

For the lower bound, we provide a strategy for Painter to win on $\mathcal{S}_{d-1}$, where d is the claimed bound. Call a vertex *saturated in color i* when it lies on $k_i - 1$ edges of color i. Painter's strategy is straightforward: when Builder presents an edge, Painter colors it with any color in which neither endpoint is already saturated.

To show that this is always possible, consider the possibility of Builder playing an edge uv. If no color is available for use on uv, then for each i, either u or v is saturated in color i. Let X be the set of colors in which u is saturated; u has degree at least $\sum_{i\in X}(k_i-1)$. Likewise, since v is saturated in the remaining colors, v has degree at least $\sum_{i\notin X}(k_i-1)$. Thus u or v already has degree at least $d-1$, and Builder cannot present uv. □

The lower bound in Theorem 2.8 yields a general lower bound:

Corollary 2.9. $\mathring{R}_\Delta(G_1, \ldots, G_s) \ge 1+\min_{X\subseteq\{1,\ldots,s\}} \max\left\{\sum_{i\in X}(k_i-1), \sum_{i\notin X}(k_i-1)\right\}$, *where $k_i = \Delta(G_i)$ for $1 \le i \le s$.*

Proof. The on-line degree Ramsey number is *monotone*: if $G'_i \subseteq G_i$ for $1 \le i \le s$, then $\mathring{R}_\Delta(G_1, \ldots, G_s) \ge \mathring{R}_\Delta(G'_1, \ldots, G'_s)$. Consequently, $\mathring{R}_\Delta(G_1, \ldots, G_s) \ge \mathring{R}_\Delta(K_{1,k_1}, \ldots, K_{1,k_s})$, and Theorem 2.8 applies. □

We next turn to general trees. Corollary 2.7 shows that $\mathring{R}_\Delta(S_{b,b}; s) = s(b-1)+1$; in fact, this is the maximum value of $\mathring{R}_\Delta(T; s)$ over all trees with maximum degree b. This was shown in [3] for $s = 2$; we prove the general result by a different approach.

Theorem 2.10. *If $T_1, \ldots, T_s$ are trees, then $\mathring{R}_\Delta(T_1, \ldots, T_s) \le \sum_{i=1}^{s}(\Delta(T_i) - 1) + 1$. Moreover, the bound holds with equality whenever all T_i have adjacent vertices of maximum degree.*

Proof. The sharpness follows from Corollary 2.7 and the monotonicity of $\mathring{R}_\Delta$.

For the upper bound, we provide a strategy for Builder. To simplify notation, let d be the claimed bound, let $k_i = \Delta(T_i)$, and let $h_i = |V(T_i)|$. If each k_i is 1, then Builder wins by presenting a single edge. We proceed by induction on $\sum_i k_i$. If any k_i is 1, then T_i is a single edge, so color i may be ignored: if Painter ever uses that color, then Builder wins. Thus Builder wins by following a strategy to win $(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_s; \mathcal{S}_d)$, the existence of which is guaranteed by the induction hypothesis.

Hence we may assume $k_i \ge 2$ for each i. Let $T^{k,h}$ denote the rooted tree in which all non-leaves have degree k and all leaves lie at distance h from the root. Since $T_i \subseteq T^{k_i,h_i}$ for each i, by monotonicity it suffices to show that $\mathring{R}_\Delta(T^{k_1,h_1}, \ldots, T^{k_s,h_s}) \le \sum_i(k_i - 1) + 1$.

Builder aims to grow a tree containing T^{k_i,h_i} in color i, for some i. More generally, let a (k,h)-*subtree* be a rooted tree with the property that all non-leaves within distance h of the root have degree k in the tree and all leaves within distance h of the root have global degree 1. Builder can force a (k_i, h_i)-subtree in color i for some i by playing a star with up to d edges. The Pigeonhole Principle yields a star with k_i edges in color i for some i by the time this is finished, and such a star is a (k_i, h_i)-subtree.

It now suffices to show that if Builder can force a (k_i, h_i)-subtree T in color i, then he can either win or force a (k_i, h_i)-subtree T' in color i that has more vertices than T within distance h_i of the root. This completes the proof because the number of vertices within distance h_i of the root of a (k_i, h_i)-subtree is maximized when the tree contains T^{k_i,h_i}.

Without loss of generality, we may assume that the monochromatic star produced by Painter when Builder starts the process with an isolated star has color 1, which we call *red*.

Let v be the root of the current red (k_1, h_1)-subtree, T. If T has no leaves with distance less than h_1 from v, then T contains T^{k_1,h_1}, and Builder wins. Otherwise, let x be some such leaf. Builder forces many copies of T (Builder plays against a consistent Painter). We consider two cases.

Case 1: $k_1 \ge 3$. By the induction hypothesis, Builder has a strategy to win the game $(T^{k_1-1,h_1}, T^{k_2,h_2}, \ldots, T^{k_s,h_s}; \mathcal{S}_{d-1})$; Builder plays this strategy on the copies of x within the copies of T. Since each copy of x had global degree 1 when its copy of T was created, the presented graph remains within $\mathcal{S}_d$. Builder either wins the original game (if the threshold is reached in another color) or forces a red T^{k_1-1,h_1} (see Figure 1, where $k_1 = 3$ and $h_1 = 2$).

In the latter case, let $\hat{T}$ be the red copy of T^{k_1-1,h_1} produced, and let x' be its root. Let v' be the copy of v within the copy of T containing x', and let T' be the maximal red tree

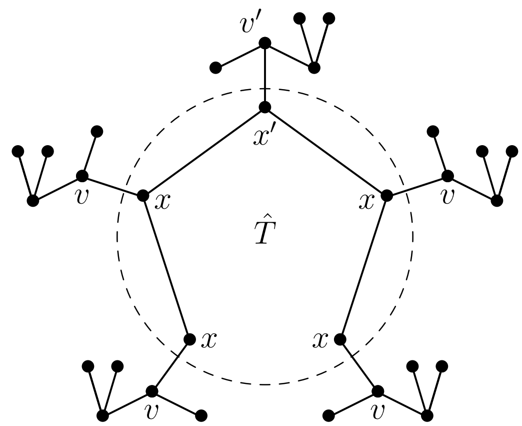

Figure 1: Induction step for Case 1 of Theorem 2.10 ($k_1 = 3$ and $h_1 = 2$)

rooted at v' (in Figure 1, all edges drawn belong to T'). All non-leaves of T' within distance h_1 of v' lie on k_1 red edges: those that were leaves in their copies of T lie on $k_1 - 1$ red edges from $\hat{T}$ and one from T, while all others were non-leaves in their copies of T. Since leaves of $\hat{T}$ lie at distance h_1 from x', their distances from v' exceed h_1, so their degrees in red are unimportant. Every leaf of T' within distance h_1 of v' has global degree 1, because each corresponds to a leaf in its copy of T. Note that T' has more vertices within distance h_1 of v' than T has within distance h_1 of its root, since x' acquires children in $\hat{T}$. Thus T' with root v' is the desired (k_1, h_1)-subtree.

Case 2: $k_1 = 2$. Builder cannot proceed as before, because T^{1,h_1} may not be well-defined. Note that T^{2,h_1} is a path. Since $k_1 = 2$, in the initial phase Builder can force many red copies of P_3 whose endpoints have global degree 1. Builder plays a winning strategy for $(P_2, T^{k_2,h_2}, \ldots, T^{k_s,h_s}; \mathcal{S}_{d-1})$ on copies of x, where x is a leaf of the current red path that Builder can force (with endpoints of global degree 1. If Painter uses no red edges, then Builder wins, by the induction hypothesis. Otherwise, Builder obtains a longer red path in which both endpoints have global degree 1; he then chooses x to be one of these endpoints and repeats the process as needed, eventually either winning or obtaining a red T^{2,h_1}. □

In the diagonal case, the bound reduces to a simpler expression:

Corollary 2.11. *If T is a tree, then $\mathring{R}_\Delta(T; s) \leq s(\Delta(T) - 1) + 1$.* □

For the sake of comparison between $\mathring{R}_\Delta$ and R_Δ, we remark that it was shown in [7] that $R_\Delta(T) \leq 2s(\Delta(T) - 1)$ for any tree T; it was shown in [9] that this bound is asymptotically tight. Thus the maximum value of the on-line degree Ramsey number over the class of trees is about half that of the "off-line" degree Ramsey number.

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