Game matching number of graphs

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Abstract

We study a competitive optimization version of $\alpha'(G)$, the maximum size of a matching in a graph G. Players alternate adding edges of G to a matching until it becomes a maximal matching. One player (Max) wants that matching to be large; the other (Min) wants it to be small. The resulting sizes under optimal play when Max or Min starts are denoted $\alpha'_g(G)$ and $\hat{\alpha}'_g(G)$, respectively. We show that always $\left|\alpha'_g(G) - \hat{\alpha}'_g(G)\right| \le 1$. We obtain a sufficient condition for $\alpha'_g(G) = \alpha'(G)$ that is preserved under cartesian product. In general, $\alpha'_g(G) \ge \frac{2}{3}\alpha'(G)$, with equality for many split graphs, while $\alpha'_g(G) \ge \frac{3}{4}\alpha'(G)$ when G is a forest. Whenever G is a 3-regular n-vertex connected graph, $\alpha'_g(G) \ge n/3$, and there are such examples with $\alpha'_g(G) \le 7n/18$. For an n-vertex path or cycle, the answer is roughly $n/7$.

1 Introduction

The archetypal question in extremal graph theory is "how many edges can an n-vertex graph contain without also containing a copy of a forbidden subgraph F? The answer is the *extremal number* of F, denoted $\mathrm{ex}(F; n)$. The celebrated theorem of Turán [16] gives a formula for $\mathrm{ex}(K_r; n)$ and characterizes the largest graphs not containing K_r.

A graph H is *F-saturated* if $F \not\subseteq H$, but $F \subseteq H + e$ whenever $e \in E(\overline{H})$. Thus the extremal number $\mathrm{ex}(F; n)$ is the maximum size of an F-saturated n-vertex graph. The *saturation number* of F, denoted $\mathrm{sat}(F; n)$, is the minimum size of an F-saturated n-vertex graph. Erdős, Hajnal, and Moon [5] initiated the study of graph saturation, determining $\mathrm{sat}(K_r; n)$.

More generally, for fixed graphs F and G, a subgraph H of G is *F-saturated relative to* G if $F \not\subseteq H$, but $F \subseteq H + e$ whenever $e \in E(G) - E(H)$. If edges are successively added to

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H without ever producing a copy of F, then the final number of edges must be between the minimum and maximum sizes of subgraphs that are F-saturated relative to G. When the edges are chosen by two players with opposing objectives, we obtain a natural game.

Definition 1.1. The *F-saturation game* on a graph G is played by two players, Max and Min. The players jointly construct a subgraph H of G by iteratively adding edges of G to H without completing a copy of F. The game ends when H is F-saturated relative to G. Max aims to maximize the length of the game, while Min aims to minimize it. The *game F-saturation number* of G is the length of the game when both players play optimally, denoted by $\mathrm{sat}_g(F, G)$ when Max starts and by $\widehat{\mathrm{sat}}_g(F, G)$ when Min starts.

The saturation game bears similarities to other games. In a *Maker-Breaker* game on a graph G (typically K_n), the players choose edges of G in turn. Maker wins by claiming the edges in a subgraph having a desired property $\mathcal{P}$; Breaker wins by preventing this. For example, Hefetz, Krivelevich, Stojaković, and Szabó [8] studied Maker-Breaker games on K_n in which Maker seeks to build non-planar graphs, non-k-colorable graphs, or K_t-minors.

In a Maker-Breaker game, one may ask how many turns Maker needs to win. Breaker then acts like Max in the saturation game: both aim to prolong the game. Hefetz, Krivelevich, Stojaković, and Szabó [9] showed that on K_n, Maker can build a spanning cycle within $n + 2$ turns and can build a k-connected subgraph within $(1 + o(1))kn/2$ turns. Feldheim and Krivelevich [6] showed that Maker can build a given d-degenerate p-vertex graph within $d^{11}2^{2d+7}p$ moves (when n is sufficiently large relative to d and p).

The saturation game was introduced by Füredi, Reimer, and Seress [7]. Calling it "a variant of Hajnal's triangle-free game", they studied $\mathrm{sat}_g(K_3, K_n)$. (In Hajnal's original "triangle-free game", the players wanted to avoid triangles, with the loser being the player forced to create one; see [4, 11, 12, 14, 15].) Since the F-saturation game always produces an F-saturated graph, trivially $n - 1 = \mathrm{sat}(K_3; n) \le \mathrm{sat}_g(K_3, K_n) \le \mathrm{ex}(K_3; n) = \lfloor n^2/4 \rfloor$. The lower bound from [7] is $\mathrm{sat}_g(K_3, K_n) \ge \Omega(n \lg n)$. As an upper bound, an unpublished result of Erdős states $\mathrm{sat}_g(K_3, K_n) \le n^2/5$. Biró, Horn, and Wildstrom [2] improved the leading coefficient in the upper bound, but the order of growth remains unknown.

In this paper, we study the P_3-saturation game. Every P_3-saturated subgraph of a graph G is a maximal matching in G; hence we call $\mathrm{sat}_g(P_3, G)$ the *game matching number of G*. With $\alpha'(G)$ denoting the maximum size of a matching in G, we let $\alpha'_g(G)$ and $\hat{\alpha}'_g(G)$ denote the sizes of the matchings produced under optimal play when Max plays first and when Min plays first, respectively.

The outcome of an F-saturation game may depend heavily on which player starts. For example, if G arises from $K_{1,k}$ by subdividing one edge, then $\mathrm{sat}_g(G, 2K_2) = k$, but $\widehat{\mathrm{sat}}_g(G, 2K_2) = 2$. The special case of game matching ($F = P_3$) is much better behaved; the main result of Section 2 states that always $\left|\alpha'_g(G) - \hat{\alpha}'_g(G)\right| \le 1$. A closely related result is that $\alpha'_g(H) \le \alpha'_g(G)$ and $\hat{\alpha}'_g(H) \le \hat{\alpha}'_g(G)$ when H is an induced subgraph of G.

Section 3 examines the relationship between $\alpha'_g(G)$ and $\alpha'(G)$. We obtain a condition on G that forces $\alpha'_g(G) = \alpha'(G)$; it is somewhat technical but is preserved by taking the cartesian product with any other graph. We also show that $\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$ for all G. This inequality is sharp; we demonstrate equality for many split graphs (a graph is a *split graph* if its vertex set can be covered by one clique and one independent set). We also determine the maximum number of edges in an n-vertex graph such that $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$.

In Section 4, we restrict our attention to forests. When G is a forest, $\alpha'_g(G) \geq \frac{3}{4}\alpha'(G)$, which is sharp (equality holds for the "comb" obtained by adding a pendant edge at each vertex of the path P_{4k}.) We also prove that $\hat{\alpha}'_g(F) \leq \alpha'_g(F)$ when F is a forest. This in turn implies that adding a star component to a forest increases both parameters by 1.

Finally, in Section 5 we consider graphs with small maximum degree. We show that optimal play on the n-vertex path P_n produces a maximal matching of size differing from $n/7$ by at most a small constant (Wise and Yeager [17] gave a more detailed answer for disjoint unions of paths), which also yields the answer for cycles. For a 3-regular n-vertex graph G, we prove that always $\alpha'_g(G) \geq n/3$, and we construct such graphs with $\alpha'_g(G) \leq 7n/18$.

2 Max-start vs. Min-start

When Max starts, we refer to the matching game as the *Max-start game*; when Min starts, we call it the *Min-start game*. Our primary goal in this section is to determine all pairs of integers (r, s) that are realizable as $(\alpha'_g(G), \hat{\alpha}'_g(G))$ for some graph G. We show first that all pairs with $|r - s| \leq 1$ are realizable. The main point is that these are the only such pairs; the choice of the starting player makes little difference. As part of the proof, we will show that for both the Max-start and Min-start games, the value is nonincreasing under taking of induced subgraphs.

Proposition 2.1. *A pair (r, s) of positive integers is realizable as $(\alpha'_g(G), \hat{\alpha}'_g(G))$ for some graph G if $|r - s| \leq 1$ (except for $(r, s) = (1, 2)$), and G may be required to be connected.*

Proof. If Min cannot move without leaving another move, then the same is true for a first move by Max, so $(1, 2)$ is not realizable. The complete graph K_{2r} realizes (r, r). To realize $(r, r - 1)$, add a pendant vertex to K_{2r-1}.

To realize $(2k, 2k + 1)$, take K_{4k+2} and discard a perfect matching. For $k = 1$, this holds by a short case analysis. For $k > 2$, any move by the first player joins endpoints of two deleted edges, and the second player can play the edge joining the other two endpoints of those edges to reach the same situation in a smaller graph.

To realize $(2k-1, 2k)$ for $k \geq 2$, take $2K_{2k}$, delete one edge from each component, creating four *special* vertices, and restore regularity by adding a different pair of edges on the special vertices. If Min makes a first move involving a special vertex, then Max can move to leave

$2K_{2k-2}$. If Max makes a first move involving a special vertex, then Min can move to leave $K_{2k-3}+K_{2k-1}$ (we use "+" to denote disjoint union). A first move not involving a special vertex can be mirrored on the vertices of the other large clique to reach the same situation in a smaller graph, except that when $k=2$ such a first move by Max can be answered by Min to leave P_3+K_1. ∎

The pathology $\alpha'_g(G) < \hat{\alpha}'_g(G)$ cannot occur when G is a forest; we prove this in Section 4. Meanwhile, we begin the proof of the general bounds $|\alpha'_g(G) - \hat{\alpha}'_g(G)| \le 1$ with a simple observation that will also be useful in other contexts.

Proposition 2.2. *If uv is an edge in a graph G, then $\alpha'_g(G) \ge 1 + \hat{\alpha}'_g(G - \{u,v\})$, with equality if and only if uv is an optimal first move for Max on G. Likewise, $\hat{\alpha}'_g(G) \le 1 + \alpha'_g(G - \{u,v\})$, with equality if and only if uv is an optimal first move for Min on G.*

Proof. The game consists of the move uv plus play on $G - \{u,v\}$ with the other player starting. By definition, continuing with optimal play on $G - \{u,v\}$ yields the optimal result for G if the initial move is an optimal move. Otherwise, optimal play by the first player achieves more in the desired direction. ∎

We facilitate the inductive proof of $|\alpha'_g(G) - \hat{\alpha}'_g(G)| \le 1$ by proving simultaneously that both game matching numbers are monotone under the deletion of vertices.

Theorem 2.3. *If G is a graph, and v is a vertex in G, then*
(1) $|\alpha'_g(G) - \hat{\alpha}'_g(G)| \le 1$, and
(2) $\alpha'_g(G) \ge \alpha'_g(G - v)$ and $\hat{\alpha}'_g(G) \ge \hat{\alpha}'_g(G - v)$.

Proof. We prove both statements simultaneously by induction on $|V(G)|$. They hold by inspection for $|V(G)| \le 2$, so consider larger G.

Step 1: *If (1) and (2) hold for smaller graphs, then (2) holds for G.* First consider $\alpha'_g(G)$. Let $H = G - v$. Let xy be an optimal first move in the Max-start game on H, and let $H' = H - \{x,y\}$. Note that $xy \in E(G)$; let $G' = G - \{x,y\}$. By Proposition 2.2, $\alpha'_g(H) = 1 + \hat{\alpha}'_g(H')$ and $\alpha'_g(G) \ge 1 + \hat{\alpha}'_g(G')$. Since $H' = G' - v$, applying (2) for G' yields

$$\alpha'_g(G) \ge 1 + \hat{\alpha}'_g(G') \ge 1 + \hat{\alpha}'_g(H') = \alpha'_g(H).$$

Now consider $\hat{\alpha}'_g(G)$; again let $H = G - v$. Let xy be an optimal first move in the Min-start game on G, and let $G' = G - \{x,y\}$. We consider two cases.

Case 1: $v \notin \{x,y\}$. Here xy can be played in the Min-start game on H; let $H' = H - \{x,y\}$. By Proposition 2.2, $\hat{\alpha}'_g(G) = 1 + \alpha'_g(G')$ and $\hat{\alpha}'_g(H) \le 1 + \alpha'_g(H')$. Since $H' = G' - v$, applying (2) for G' yields

$$\hat{\alpha}'_g(G) = 1 + \alpha'_g(G') \ge 1 + \alpha'_g(H') \ge \hat{\alpha}'_g(H).$$

Case 2: $v \in \{x, y\}$. We may assume $v = x$. If y is isolated in H, then the irrelevance of isolated vertices and (1) for H yield

$$\hat{\alpha}'_g(G) = 1 + \alpha'_g(G') = 1 + \alpha'_g(H - y) = 1 + \alpha'_g(H) \geq \hat{\alpha}'_g(H).$$

If instead y has a neighbor z in H, then yz can be played in the Min-start game on H; let $H' = H - \{y, z\}$. By the optimality of xy and Proposition 2.2, $\hat{\alpha}'_g(G) = 1 + \alpha'_g(G')$ and $\hat{\alpha}'_g(H) \leq 1 + \alpha'_g(H')$. Since $H' = G' - z$, applying (2) for G' yields

$$\hat{\alpha}'_g(G) = 1 + \alpha'_g(G') \geq 1 + \alpha'_g(H') \geq \hat{\alpha}'_g(H).$$

Step 2: *If (2) holds for G (and smaller graphs), then (1) holds for G.* To prove $\hat{\alpha}'_g(G) \leq 1+\alpha'_g(G)$, let uv be an optimal first move in the Min-start game on G, and let $G' = G-\{u, v\}$. Applying (2) to both $G - v$ and G yields

$$\hat{\alpha}'_g(G) = 1 + \alpha'_g(G') \leq 1 + \alpha'_g(G - v) \leq 1 + \alpha'_g(G).$$

The inequality $\alpha'_g(G) \leq 1 + \hat{\alpha}'_g(G)$ follows similarly. ∎

As a corollary, we obtain the following result:

Corollary 2.4. *If v is a vertex in a graph G, then $\alpha'_g(G) \geq \alpha'_g(G - v) \geq \alpha'_g(G) - 2$ and $\hat{\alpha}'_g(G) \geq \hat{\alpha}'_g(G - v) \geq \hat{\alpha}'_g(G) - 2$, and the bounds are sharp.*

Proof. The upper bounds for $G - v$ are from Theorem 2.3. The lower bounds hold when v is isolated because isolated vertices do not affect play. Otherwise, let u be a neighbor of v. By Proposition 2.2 and Theorem 2.3(1), $\hat{\alpha}'_g(G) \leq 1 + \alpha'_g(G - \{u, v\}) \leq 2 + \hat{\alpha}'_g(G - \{u, v\})$, and then $\hat{\alpha}'_g(G-v) \geq \hat{\alpha}'_g(G-\{u, v\}) \geq \hat{\alpha}'_g(G)-2$ by Theorem 2.3(2). For the remaining inequality, Proposition 2.2 and Theorem 2.3 yield $\alpha'_g(G-v) \geq \alpha'_g(G-\{u, v\}) \geq \hat{\alpha}'_g(G)-1 \geq \alpha'_g(G)-2$.

For the sharpness of the upper bounds, equality holds when v is isolated. For an example with $\alpha'_g(G - v) = \alpha'_g(G) - 2$, let $G = K_2 + C_6$. By inspection, $\alpha'_g(G) = 4$; Max should play the isolated edge first. If v is a vertex of the isolated edge, then the desired first move is unavailable, and Max must play before Min on the 6-cycle, yielding $\alpha'_g(G - v) = 2$.

For the Min-start inequality, let $G = 2K_2 + C_6$, and again let v be a vertex of an isolated edge. Similar arguments show that now $\hat{\alpha}'_g(G) = 5$ and $\hat{\alpha}'_g(G - v) = 3$. ∎

3 Relation to Matching Number

We next study the relationship between the game matching number and the ordinary matching number. Generally speaking, Max cannot guarantee $\alpha'_g(G) = \alpha'(G)$. For example, in Section 5 we will see that $\alpha'_g(P_n)$ is about $3n/7$, even though $\alpha'(P_n) = \lfloor n/2 \rfloor$. Nevertheless, there is a condition under which the parameters coincide on graphs with perfect matchings.

Theorem 3.1. *Fix an n-vertex graph G and a perfect matching M in G. If $uw \in E(G) \Rightarrow vx \in E(G)$ whenever $uv, wx \in M$, then $\alpha'_g(G) = n/2$ and $\hat{\alpha}'_g(G) = n/2$.*

Proof. Both claims hold by inspection when $n \leq 4$; we proceed by induction on n.

Since neither $\alpha'_g(G)$ nor $\hat{\alpha}'_g(G)$ can exceed $n/2$, we need only give strategies for Max. In the Max-start game, Max first plays an edge uv in M. Now $\alpha'_g(G) \geq 1 + \hat{\alpha}'_g(G - \{u, v\})$. Moreover, $G - \{u, v\}$ satisfies the hypothesis (using $M - \{uv\}$), so the induction hypothesis yields $\alpha'_g(G) \geq 1 + (n-2)/2 = n/2$.

In the Min-start game, let uv be an optimal first move for Min. If $uv \in M$, then the graph $G - \{u, v\}$ satisfies the hypothesis (using $M - \{uv\}$), so the induction hypothesis yields $\hat{\alpha}'_g(G) = 1 + \alpha'_g(G - \{u, v\}) = 1 + (n-2)/2 = n/2$.

If instead $uv \notin M$, then since M is a perfect matching there exist edges $uu', vv' \in M$. By the hypothesis, $u'v' \in E(G)$; Max plays this edge. Now $G - \{u, v, u', v'\}$ satisfies the hypothesis (using $M - \{uu', vv'\}$), so the induction hypothesis yields

$$\hat{\alpha}'_g(G) = 1 + \alpha'_g(G - \{u, v\}) \geq 2 + \hat{\alpha}'_g(G - \{u, v, u', v'\}) = 2 + (n-4)/2 = n/2. \qquad \blacksquare$$

The property we require of G in Theorem 3.1 is somewhat restrictive. However, it is preserved under cartesian products with arbitrary graphs. That is, if G has a perfect matching M satisfying the hypothesis of Theorem 3.1, then so does $G \,\square\, H$, for any graph H (using the perfect matching in $G \,\square\, H$ formed by the copies of M). As a special case, the observation that $K_{r,r}$ has such a perfect matching yields the following:

Corollary 3.2. *For $r \geq 1$ and any graph H, Max can force a perfect matching in $K_{r,r} \,\square\, H$, no matter who plays first.*

Theorem 3.1 does not imply that Max can force a perfect matching in all graphs arising as cartesian products.

Example 3.3. Let G be the "paw", obtained from a triangle with vertex set u, v, w by adding one vertex x with neighbor w. In the graph $G \,\square\, P_3$, where P_3 is the path with vertices a, b, c in order, Max cannot force a perfect matching no matter who starts. Min can start by playing the edge joining (w, a) and (v, a), threatening to isolate either (u, a) or (x, a). Max has no move that will prevent Min from isolating one of these vertices on the next turn. Similar analysis applies to the Max-start game on $G \,\square\, P_3$ and to both games on $G \,\square\, G$.

The next natural question is: By how much can $\alpha'_g(G)$ and $\alpha'(G)$ differ? We prove a general lower bound on $\alpha'_g(G)$ in terms of $\alpha'(G)$ and then show that it is sharp. A *round* of play consists of a move by Max followed by a move by Min. We call $G - \bigcup_i \{u_i, v_i\}$ the *residual graph* after edges $u_1v_1, \ldots, u_kv_k$ are played in the matching game on G.

Theorem 3.4. *$\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$ for every graph G.*

Proof. As long as an edge remains, Max plays an edge belonging to a maximum matching; this reduces the matching number by 1. An edge played by Min is incident to at most two edges of a maximum matching and hence reduces the matching number (of the residual graph) by at most 2. Hence when $\alpha'(G) \geq 3$, a round reduces the matching number of the residual graph by at most 3 while adding 2 to the size of the matching played. Finally, when $\alpha'(G) \in \{1, 2\}$, Max can play an edge of a maximum matching, and if $\alpha'(G) = 2$ there will be another edge that Min must play. ∎

Theorem 3.4 is sharp. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. We present a Min strategy for the matching game on split graphs. On many split graphs, this strategy obtains equality in Theorem 3.4.

Proposition 3.5. *Let G be a split graph. If $V(G) = S \cup T$, with S an independent set and T a clique, then $\alpha'_g(G) \leq \left\lceil \frac{2}{3} |T| \right\rceil$.*

Proof. On each turn, Min plays an edge joining two vertices of T if possible, and any legal move otherwise. By the choice of S and T, every edge in G has at least one endpoint in T. Thus each move by Max covers at least one vertex of T, and each move by Min covers two vertices of T while two remain. Thus each round increases the size of the matching by 2 and decreases $|T|$ by at least 3, until at most two vertices remain in T; the small cases are then checked explicitly. ∎

Whenever a clique T contains an endpoint of every edge in a graph G, we have $\alpha'(G) \leq |T|$. If $\alpha'(G) = |T|$ and $|T| \equiv 0 \mod 3$, then the lower bound in Proposition 3.5 matches the upper bound in Theorem 3.4. Thus equality holds in Theorem 3.4 for this family of split graphs. We can introduce all the edges joining the clique and the independent set, as in the next example.

Example 3.6. For $k \geq 1$ and $n \geq 6k$, form G from K_n by deleting the edges of a complete subgraph with $n - 3k$ vertices; G is a split graph whose clique T has $3k$ vertices. Note that $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$. In fact, also G is $3k$-connected.

The construction in Example 3.6 maximizes the number of edges among all n-vertex graphs such that $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$. To prove this, we need a technical lemma about ordinary matching; it is a special case of a more difficult result of Brandt [3]. Since we only need this special case, we provide a short, self-contained proof.

Lemma 3.7. *If G is an n-vertex graph, then $\alpha'(G) \geq \min\{\lfloor n/2 \rfloor, \delta(G)\}$.*

Proof. Let M be a maximum matching in G. If the conclusion fails, then we may let u and v be distinct vertices not covered by M. Let a_u and b_u count the edges in M having exactly one and exactly two endpoints in $N(u)$, respectively; define a_v and b_v similarly.

By the maximality of M, all neighbors of u are covered by M. Thus $a_u + 2b_u \geq \delta(G)$. Similarly, $a_v + 2b_v \geq \delta(G)$. If $w, x \in N(u) \cap N(v)$ for some edge $wx \in M$, then replacing wx with uw and vx enlarges M. Thus no edge in M counts toward both b_u and b_v. Hence $|M| \geq \frac{1}{2}(a_u + a_v) + b_u + b_v \geq \delta(G)$. ■

The next result was observed by Plummer [13].

Lemma 3.8. *If G has n vertices, and $\delta(G) \geq \lfloor n/2 \rfloor + 1$, then every edge of G lies in a matching of size $\lfloor n/2 \rfloor$.*

Proof. For $uv \in E(G)$, let $G' = G - \{u, v\}$. Since $\delta(G') \geq \lfloor (n-2)/2 \rfloor$, Lemma 3.7 implies $\alpha'(G') \geq \lfloor (n-2)/2 \rfloor$. Replace uv. ■

Lemma 3.9. *If v is a non-isolated vertex of a graph G, then some maximum matching in G contains an edge incident to v.*

Proof. In a maximal matching, each neighbor u of an unmatched vertex v must be matched. Such an edge can be replaced by the edge uv. ■

Theorem 3.10. *If G is an n-vertex graph with $\alpha'(G) = 3k$ and $\alpha'_g(G) = 2k$, then $|E(G)| \leq \binom{3k}{2} + 3k(n - 3k)$.*

Proof. We prove the claim by induction on k. When $k = 0$, the claim is $|E(G)| \leq 0$, which holds since $\alpha'(G) = 0$ requires $E(G) = \varnothing$.

For $k \geq 1$, we seek an edge uv for Max to play such that $\alpha'(G - \{u, v\}) = \alpha'(G) - 1$ and $d(u) + d(v) \leq n - 1 + 3k$; call this a *good edge*. Playing a good edge eliminates at most $n - 2 + 3k$ edges from the residual graph. With $n - 2$ vertices remaining, the next move by Min eliminates at most $2n - 7$ more edges.

Let G' be the residual graph after Max plays a good edge uv and Min then plays an optimal move. By Proposition 2.2, $\alpha'_g(G) \geq 2 + \alpha'_g(G')$, so $\alpha'_g(G') \leq 2k - 2$. Playing uv reduces the matching number only by 1, and the move by Min reduces it by at most 2, so $\alpha'(G') \geq 3k - 3$. Now $\alpha'_g(G') = 2k - 2$ and $\alpha'(G') = 3k - 3$, by Theorem 3.4. Thus the induction hypothesis applies to G'. Adding the edges of G not in G' yields

$$|E(G)| \leq |E(G')| + 3n + 3k - 9 \leq \binom{3(k-1)}{2} + 3(k-1)(n-4 - 3(k-1)) + 3n + 3k - 9$$
$$= \frac{(3k-3)(3k-4)}{2} + 9k - 6 + 3k(n - 3k) = \binom{3k}{2} + 3k(n - 3k).$$

In most cases, G has a good edge. We may assume that G has no isolated vertices, since discarding them does not affect the matching number, the game matching number, or the number of edges, and the edge bound for the smaller graph is less than what we allow for G. If $\delta(G) \leq 3k$, then let v be a vertex of minimum degree. Lemma 3.9 implies that some maximum matching contains an edge uv incident to v, and $d(v) \leq 3k$ implies $d(u) + d(v) \leq n - 1 + 3k$. Hence uv is good.

If $n \geq 6k + 2$, then $\lfloor n/2 \rfloor > 3k$, and Lemma 3.7 implies $\delta(G) \leq 3k$. Since $\alpha'(G) = 3k$ implies $n \geq 6k$, we may henceforth assume $n \in \{6k, 6k+1\}$ and $\delta(G) \geq 3k+1 = \lfloor n/2 \rfloor + 1$.

If $\delta(G) \geq \lfloor n/2 \rfloor + 2$, then let G' be the residual graph after any move by Max. Since $\delta(G') \geq \lfloor (n-2)/2 \rfloor + 1$, Lemma 3.8 implies that every edge of G' lies in a matching of size $\lfloor (n-2)/2 \rfloor$. Thus the subsequent move by Min cannot reduce the matching number by 2. This means that the first two moves of the game produce a residual graph G'' with $\alpha'(G'') \geq 3k - 2$ and hence $\alpha'_g(G'') > 2k - 2$, so $\alpha'_g(G) > 2k$.

The remaining case is $\delta(G) = 3k + 1 = \lfloor n/2 \rfloor + 1$. Every edge lies in a matching of size $3k$ (by Lemma 3.8), so any edge with degree-sum at most $n - 1 + 3k$ is a good edge. For a vertex v of minimum degree, we conclude that every neighbor u of v has degree at least $n-1$. Since $n-1 \geq 3k+2$ when $n \geq 6k \geq 6$, we have $\delta(G - \{u, v\}) \geq 3k = \lfloor (n-2)/2 \rfloor + 1$, and again Min cannot reduce the matching number by 2 after Max plays uv. ∎

4 Forests

Although the lower bound in Theorem 3.4 is tight infinitely often, even when arbitrarily high connectivity is required, the bound can be improved on some classes of graphs. Our main result in this section improves the guarantee for Max to $\alpha'_g(F) \geq \frac{3}{4}\alpha'(F)$ when F is a forest. In order to prove it, we will also need that $\hat{\alpha}'_g(F) \leq \alpha'_g(F)$ when F is a forest.

Intuitively, $\hat{\alpha}'_g(F) \leq \alpha'_g(F)$ means that in every forest, there is no advantage to playing second rather than first. In this intuitive sense, the result is equivalent to the statement that $\hat{\alpha}'_g(F + K_{1,t}) = \hat{\alpha}'_g(F) + 1$ and $\alpha'_g(F + K_{1,t}) = \alpha'_g(F) + 1$ for every forest F and all $t \in \mathbb{N}$. In every instance of the matching game on $F + K_{1,t}$, exactly one move will be played on the added star; thus playing on the star before being forced to is equivalent to choosing to play second on the remaining forest. Each of these two claims helps us prove the other, so we prove them together by induction.

Theorem 4.1. *For every forest F and all $t \in \mathbb{N}$, the following two statements hold:*
(1) $\hat{\alpha}'_g(F) \leq \alpha'_g(F)$.
(2) $\hat{\alpha}'_g(F + K_{1,t}) = 1 + \hat{\alpha}'_g(F)$ and $\alpha'_g(F + K_{1,t}) = 1 + \alpha'_g(F)$.

Proof. We prove both statements simultaneously by induction on $|V(F)|$. They hold by inspection for $|V(F)| \leq 2$, so consider larger F.

Step 1: *If (1) and (2) hold for smaller forests, then (1) holds for F.* Let $k = \hat{\alpha}'_g(F)$. By Theorem 2.3, the claim holds unless $\alpha'_g(F) = k - 1$. When every component is a star, immediately $\hat{\alpha}'_g(F) = \alpha'_g(F)$, so we may choose a component C in F that is not a star. Since C is not a star, a longest path in C has at least four vertices; let the first three be u, v, w in order (u is a leaf).

Let $F' = F - \{v, w\}$. By Proposition 2.2, $\hat{\alpha}'_g(F) \le 1 + \alpha'_g(F')$, so $\alpha'_g(F') \ge k - 1$. Similarly, $\alpha'_g(F) \ge 1 + \hat{\alpha}'_g(F')$, so $\hat{\alpha}'_g(F') \le k - 2$. Now Theorem 2.3 requires $\alpha'_g(F') = k - 1$ and $\hat{\alpha}'_g(F') = k - 2$. By Corollary 2.4, also $\alpha'_g(F - w) = k - 1$.

Let $t = d(v)$. The component of $F - w$ containing v is $K_{1,t-1}$, so $F - w = F^* + K_{1,t-1}$ for some forest F^*. Moreover, $F^* + (t-1)K_1 = F - \{v, w\} = F'$, so $\alpha'_g(F^* + (t-1)K_1) = k - 1$. Applying (2) to the smaller forest F^* now yields the contradiction

$$k - 1 = \alpha'_g(F^* + K_{1,t-1}) = 1 + \alpha'_g(F^*) = 1 + \alpha'_g(F^* + (t-1)K_1) = k.$$

Step 2: *If (1) and (2) hold for smaller forests and (1) holds for F, then (2) holds for F.* We have four inequalities to prove. We prove them by considering an optimal first move uv in the Min-start or Max-start game on F or on $F + K_{1,t}$. In the displayed computations when uv is chosen from $F + K_{1,t}$, we use the choice of uv, the validity of (2) for F', and Proposition 2.2, in that order. When uv is an optimal first move in a game on F, we use the same three things in the reverse order. In each case $F' = F - \{u, v\}$ when $uv \in E(F)$.

2a: *uv is an optimal first move in the Min-start game on $F + K_{1,t}$.* If $uv \notin E(F)$, then $\hat{\alpha}'_g(F + K_{1,t}) = 1 + \alpha'_g(F) \ge 1 + \hat{\alpha}'_g(F)$, since (1) holds for F. Otherwise,

$$\hat{\alpha}'_g(F + K_{1,t}) = 1 + \alpha'_g(F' + K_{1,t}) = 1 + \alpha'_g(F') + 1 \ge 1 + \hat{\alpha}'_g(F).$$

2b: *uv is an optimal first move in the Min-start game on F:*

$$\hat{\alpha}'_g(F + K_{1,t}) \le 1 + \alpha'_g(F' + K_{1,t}) = 1 + \alpha'_g(F') + 1 = 1 + \hat{\alpha}'_g(F).$$

2c: *uv is an optimal first move in the Max-start game on F:*

$$\alpha'_g(F + K_{1,t}) \ge 1 + \hat{\alpha}'_g(F' + K_{1,t}) = 1 + \hat{\alpha}'_g(F') + 1 = 1 + \alpha'_g(F).$$

2d: *uv is an optimal first move in the Max-start game on $F + K_{1,t}$.* If $uv \notin E(F)$, then $\alpha'_g(F + K_{1,t}) = 1 + \hat{\alpha}'_g(F) \le 1 + \alpha'_g(F)$, since (1) holds for F. Otherwise,

$$\alpha'_g(F + K_{1,t}) = 1 + \hat{\alpha}'_g(F' + K_{1,t}) = 1 + \hat{\alpha}'_g(F') + 1 \le 1 + \alpha'_g(F). \qquad \blacksquare$$

Theorem 4.2. *If F is a forest, then $\alpha'_g(F) \ge \frac{3}{4}\alpha'(F)$.*

Proof. We prove simultaneously by induction on $|E(F)|$ that $\alpha'_g(F) \geq \frac{3}{4}\alpha'(F)$ and $\hat{\alpha}'_g(F) \geq \frac{3}{4}\alpha'(F) - \frac{1}{2}$. By Theorem 4.1, each component of F that is a star contributes 1 to each of $\alpha'(F)$, $\alpha'_g(F)$, and $\hat{\alpha}'_g(F)$, so both claims clearly hold when all components of F are stars. Moreover, it suffices to prove the claims when no components are stars.

Consider the Min-start game. Let uv be an optimal first move, and let $F' = F - \{u, v\}$. Since vertex deletion always decreases the matching number by at most 1, we have $\alpha'(F') \geq \alpha'(F) - 2$. This observation, the choice of uv, and the induction hypothesis yield

$$\hat{\alpha}'_g(F) = 1 + \alpha'_g(F') \geq 1 + \frac{3}{4}\alpha'(F') \geq 1 + \frac{3}{4}(\alpha'(F) - 2) = \frac{3}{4}\alpha'(F) - \frac{1}{2}.$$

Now consider the Max-start game. Let P be a longest path in a component C of F, starting with vertices x, w, v in order. Since C is not a star, v is not isolated in F - w,x. By Proposition 3.9, v lies on an edge in some maximum matching in F - w,x; let u be the other endpoint of such an edge. Since every maximum matching in $F - w, x$ extends to a maximum matching in F by adding wx, also uv belongs to a maximum matching in F. Max plays uv; let $F' = F - \{u, v\}$.

Note that $\alpha'(F') = \alpha'(F) - 1$. Also, the component of F' containing wx is a star. Hence F' has the form $F^* + K_{1,t}$, for some forest F^* and $t \in \mathbb{N}$. By Proposition 2.2, Theorem 4.1, and the induction hypothesis,

$$\alpha'_g(F) \geq 1 + \hat{\alpha}'_g(F') = 1 + \hat{\alpha}'_g(F^* + K_{1,t}) = 2 + \hat{\alpha}'_g(F^*) \geq 2 + \frac{3}{4}\alpha'(F^*) - \frac{1}{2}.$$

Since uv lies in a maximum matching in F,

$$\alpha'(F) = \alpha'(F') + 1 = \alpha'(F^* + K_{1,t}) + 1 = \alpha'(F^*) + 2.$$

These observations together yield

$$\alpha'_g(F) \geq 2 + \frac{3}{4}\alpha'(F^*) - \frac{1}{2} = \frac{3}{2} + \frac{3}{4}(\alpha'(F) - 2) = \frac{3}{4}\alpha'(F).$$ ∎

The bound $\alpha'_g(F) \geq \frac{3}{4}\alpha'(F)$ is sharp. A *comb* is a tree obtained from a path by attaching a pendant leaf to each vertex; we denote by B_ℓ the comb with ℓ leaves (and 2ℓ vertices).

Theorem 4.3. *$\alpha'_g(B_{4k}) = \frac{3}{4}\alpha'(B_{4k})$, for $k \in \mathbb{N}$.*

Proof. Since $\alpha'(B_{4k}) = 4k$, we must show that $\alpha'_g(B_{4k}) = 3k$. The lower bound follows from Theorem 4.2. To prove $\alpha'_g(B_{4k}) \leq 3k$, we give a strategy for Min.

At any point in the game on B_{4k}, the residual graph is a disjoint union of combs and isolated vertices. A played edge joining non-leaf vertices is a *Type A move*; a *Type B move* is a played edge incident to a leaf. In analyzing the strategy, we consider the number of non-isolated vertices remaining. Each Type A move reduces this number by 4 (the endpoints of the played edge and their neighboring leaves), and each Type B move reduces it by 2.

On each turn, Min chooses a largest component in the residual graph. If this component C is an isolated edge, then Min plays that edge. Otherwise, the non-leaf vertices of C induce a path, and Min plays an end-edge of that path. By these rules, Min plays only Type A moves until the residual graph consists only of isolated edges and isolated vertices.

To conclude $\alpha'_g(B_{4k}) \le 3k$, we show that at least $2k$ vertices become isolated during the game; they must remain unmatched, so the resulting matching has at most $(1/2)(8k-2k)$ edges. Each Type A move isolates two vertices, while each Type B move isolates none. It suffices to show that if there have not yet been k Type A moves, and not yet $2k$ moves altogether, then Min will make a Type A move on the next turn.

Since Min plays Type A moves at the "end" of a comb, Min never increases the number of nontrivial components in the residual graph. On the other hand, each move by Max increases that number by at most 1. Thus, if there have not yet been k Type A moves, then the number of nontrivial components at present is at most $k+1$. Let s be the number of nonisolated vertices in the residual graph before Min moves. If there have been a Type A moves and b Type B moves, then $a \le k-1$ and $a+b \le 2k-1$ yield

$$s = 8k - 4a - 2b = 8k - 2a - 2(a+b) \ge 2k + 4.$$

Hence the average number of vertices per nontrivial component in the residual graph is at least $(2k+4)/(k+1)$, which exceeds 2. Thus some component in the residual graph has more than two vertices, and Min makes another Type A move. ∎

5 Graphs with Small Maximum Degree

The following corollary of Theorem 4.1 is sometimes useful.

Corollary 5.1. *For a forest F and an integer t, the starting player has an optimal first move in F in both the Max-start and Min-start games on $F + K_{1,t}$.*

Proof. Consider the Min-start game; the proof for the other is similar. Let uv be an optimal first move in the Min-start game on F, and let $F' = F - \{u, v\}$. If Min plays an edge from the added star in the first move of the Min-start game on $F + K_{1,t}$, and play is optimal thereafter, then the length of the game will be $1 + \alpha'_g(F)$. If Min first plays uv, then the length of the game will be $1 + \alpha'_g(F' + K_{1,t})$. By Theorem 4.1, Proposition 2.2, and Theorem 4.1,

$$1 + \alpha'_g(F' + K_{1,t}) = 2 + \alpha'_g(F') = 1 + \hat{\alpha}'_g(F) \le 1 + \alpha'_g(F).$$

Hence Min does no worse by playing uv than by playing in the added star. ∎

As an application of Corollary 5.1, we determine the asymptotic value of $\alpha'_g(P_n)$. While the corollary is not strictly needed to prove this result, it simplifies the argument.

Theorem 5.2. *For all n, we have $3\left\lfloor \frac{n}{7} \right\rfloor \le \alpha'_g(P_n) \le 3\left\lceil \frac{n}{7} \right\rceil$.*

Proof. We claim that $\alpha'_g(P_{7k}) = 3k$, from which the desired result follows by Corollary **??**.

At each point during a game on P_{7k}, the residual graph is a disjoint union of paths. Each move increases the number of components in the residual graph by at most 1.

Upper bound: We give a strategy for Min. On each turn, Min selects a largest remaining component. If it has at least three vertices, then Min plays the second edge of this component. Otherwise, every component is an isolated vertex or edge; Min plays an isolated edge or the game is over. Let t be the number of turns played when the second phase begins, with s_1 isolated vertices and s_2 isolated edges.

There are at most $t+1$ components at that time, so $s_1 + s_2 \le t + 1$. Since each move deletes two vertices, $s_1 + 2s_2 + 2t = 7k$. Now

$$7k + s_1 = 2t + 2s_2 + 2s_1 \le 2t + 2(t+1) = 4t + 2.$$

Since each move by Min in the first phase isolates at least one vertex,

$$(t-1)/2 \le s_1 \le 4t + 2 - 7k.$$

Simplifying yields $t \ge \lceil (14k-5)/7 \rceil = 2k$. Moreover, $s_1 \ge \lceil (t-1)/2 \rceil \ge k$. Since exactly s_1 vertices remain unmatched at the end of the game, $\alpha'_g(P_{7k}) \le (n - s_1)/2 \le 3k$.

Lower bound: We give a strategy for Max. On each turn, Max selects a largest remaining component. If it has at least four vertices, then Max plays the third edge of this component. Otherwise, Max plays any edge in the selected component. By Corollary 5.1, we may assume that Min does not play an isolated edge unless no other moves remain.

Let t be the number of turns played when the second phase begins, with s_i remaining components having i vertices, for $i \le 3$. Since each previous move by Max created an isolated edge, and neither player has yet played an isolated edge, $s_2 \ge t/2$. Each move has increased the number of remaining components by 1, so $s_1 + s_2 + s_3 \le t + 1$. Thus $s_1 + s_3 \le t/2 + 1$.

Counting the vertices played and the vertices remaining yields

$$7k = 2t + s_1 + 2s_2 + 3s_3 \ge 2t + (s_1 + s_3) + 2s_2 \ge 3t + (s_1 + s_3),$$

so $s_1 + s_3 \le 7k - 3t$. Taking $1/7$ times this inequality plus $6/7$ times $s_1 + s_3 \le t/2 + 1$ yields $s_1 + s_3 \le k + 6/7$. By integrality, $s_1 + s_3 \le k$. At the end of the game, exactly $s_1 + s_3$ vertices remain unmatched, so $\alpha'_g(P_{7k}) \ge 3k$. ∎

Wise and Yeager [17] independently determined the exact value by the congruence class of n modulo 7 and studied the game on disjoint unions of paths. Their arguments require more careful analyses, but use essentially the same strategies as in the proof of Theorem 5.2.

Our final results concern regular graphs. When G is 3-regular, $\alpha'(G) \geq \lceil 4(|V(G)| - 1)/9 \rceil$, and this is sharp [1]. We seek an analogous sharp lower bound for $\alpha'_g(G)$ in terms of $|V(G)|$. Using the bound for $\alpha'(G)$, Theorem 3.4 yields approximately $\alpha'(G) \geq \frac{8}{27}|V(G)|$. An easy argument proves a stronger bound, which we phrase for general regular graphs.

Proposition 5.3. *If G is a connected n-vertex r-regular graph, then $\alpha'_g(G) \geq \frac{rn-2}{4r-3}$.*

Proof. We give a strategy for Max to ensure that edges are removed from the residual graph "slowly". Max first plays any edge. On each turn thereafter, Max plays any edge incident to a vertex of smallest nonzero degree.

Since G is connected, the residual graph at any time after the first move has a vertex of nonzero degree less than r. Thus each move by Max after the first deletes at most $2r - 2$ edges. Each move by Min deletes at most $2r - 1$ edges. Since Max moves before Min, after k turns the residual graph contains at least $\frac{r}{2}n - \left\lfloor \frac{4r-3}{2}k \right\rfloor - 1$ edges. To end the game, all edges must be deleted. At that time $k \geq \frac{rn-2}{4r-3}$. ∎

The bound in Proposition 5.3 is not sharp. For $r \geq 5$, this bound is weaker than the one obtained from Theorem 3.4 by using the best-known lower bounds on $\alpha'(G)$. For odd r, Henning and Yeo [10] proved that $\alpha'(G) \geq \frac{(r^3-r^2-2)n-2r+2}{2r^3-6r}$, and this is sharp. The coefficient on the linear term tends to 1/2. Multiplied by 2/3 from Theorem 3.4, the lower bound on $\alpha'_g(G)$ would be asymptotic to $n/3$, while the lower bound from Proposition 5.3 is asymptotic only to $n/4$.

For $r = 3$, Proposition 5.3 yields $\alpha'_g(G) \geq \frac{n}{3} - \frac{2}{9}$, which also is not sharp. It is easy but tedious to improve the lower bound to $n/3$ by considering the end of the game. If Min moves last and kills five remaining edges, then the play by Min is the center of a double star, and the previous move by Max could not have reduced all four leaves to degree 1. Hence a vertex of degree 1 was available to Max, and Max would have deleted only three edges instead of four. This implies that the last two moves delete at most eight edges. Case analysis along these lines improves the additive constant to 0. We omit this, partly because we also do not think that the coefficient on the linear term is sharp.

In the other direction, we provide a construction where the coefficient is 7/18. Let K be the five-vertex graph formed by subdividing one edge of K_4. Let T_k denote the *complete cubic tree of height k*, the rooted tree in which each non-leaf vertex has degree 3 and each leaf has distance k from the root. By growing two edges from a leaf, if follows by induction that in a tree where every vertex has degree 1 or 3, the number of leaves exceeds the number of non-leaf vertices by 2.

Theorem 5.4. *For $k \geq 1$, if G_k is the cubic graph obtained from T_k by making each leaf adjacent to two copies of K (via their vertices of degree 2), then $\alpha'_g(G_k) \leq (7/18)\,|V(G_k)|$.*

Proof. The copy of T_k contains $3 \cdot 2^{i-1}$ vertices at distance i from the root, for $1 \leq i \leq k$. The number of leaves is $3 \cdot 2^{k-1}$, and the number of non-leaves is $3 \cdot 2^{k-1} - 2$. Thus

$$|V(G_k)| = |V(T_k)| + 10(3 \cdot 2^{k-1}) = 12(3 \cdot 2^{k-1}) - 2 = 18 \cdot 2^k - 2.$$

We give a strategy for Min in the game on G_k. Let B be the graph consisting of K plus a pendant vertex neighboring the vertex of degree 2 in K. View G_k as the union of T_k and $6 \cdot 2^{k-1}$ copies of B. Min plays so that at most two edges are played in each copy of B.

Let uv be the previous move by Max. If uv belongs to some copy of B in which no other edge has been played, then Min plays in that copy of B to isolate at least one vertex. Min may not be able to use the edge incident to the pendant vertex, since that vertex lies on two other edges of T_k, one of which may have been played earlier; nevertheless, it follows by inspection that Min can always find a move of the desired type.

Suppose uv is otherwise. If possible, Min chooses a copy of B in which no edge has been played and plays any of the three edges that lies in no perfect matching of B. If already some edge has been played in every copy of B, then Min plays any legal move.

By construction, Min ensures that at most two edges are played in each copy of B. To bound the number of edges played in T_k, we bound $\alpha'(T_k)$. Let S be the set of vertices in T_k whose distance from a nearest leaf is odd. Note that $T_k - S$ has no edges, so $\alpha'(T_k) \leq |S|$. Also, the number of vertices at distance i from the root is at most half the number of vertices at distance $i+1$; hence $|S| \leq (1/3)\,|V(T_k)|$. Thus

$$\alpha'(T_k) \leq \lfloor (1/3)(3 \cdot 2^{k-1} + 3 \cdot 2^{k-1} - 2) \rfloor = 2^k - 1.$$

It now follows that

$$\alpha'_g(G_k) \leq 2^k - 1 + 6 \cdot 2^{k-1} \cdot 2 = 7 \cdot 2^k - 1 \leq \frac{7}{18}(18 \cdot 2^k - 2). \qquad \blacksquare$$

References

[1] T. Biedl, E.D. Demaine, C.A. Duncan, R. Fleischer, and S.G. Kobourov, Tight bounds on maximal and maximum matchings, *Discr. Math.* 285 (2004), 7–15.

[2] C. Biró, P. Horn, and J. Wildstrom, personal communication.

[3] S. Brandt, Subtrees and subforests of graphs, *J. Combin. Theory B* 61 (1994), no. 1, 63–70.

[4] S. C. Cater, F. Harary, and R. W. Robinson, One-color triangle avoidance games, in *Proc. 32nd SE Intl. Conf. Combin., Graph Theory and Comput. (Baton Rouge, 2001)*, *Congr. Numer.* 153 (2001), 211–221.

[5] P. Erdős, A. Hajnal, and J. W. Moon, A problem in graph theory, *Amer. Math. Monthly* 71 (1964), 1107–1110.

[6] O. N. Feldheim and M. Krivelevich, Winning fast in sparse graph construction games, *Combin. Probab. Comput.* 17 (2008), no. 6, 781–791.

[7] Z. Füredi, D. Reimer, and A. Seress, Hajnal's triangle-free game and extremal graph problems, in *Proc. 22nd SE Intl. Conf. Combin., Graph Theory and Comput. (Baton Rouge, 1991)*, *Congr. Numer.* 82 (1991), 123–128.

[8] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, Planarity, colorability, and minor games, *SIAM J. Discr. Math.* 22 (2008), no. 1, 194–212.

[9] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, Fast winning strategies in avoider-enforcer games, *Graphs Combin.* 25 (2009), no. 4, 533–544.

[10] M.A. Henning and A. Yeo, Tight lower bounds on the size of a maximum matching in a regular graph, *Graphs Combin.* 23 (2007), 647–657.

[11] N. Mehta and A. Seress, Connected, bounded degree, triangle avoidance games, *Electron. J. Combin.* 18 (2011), no. 1, Paper 193.

[12] N. Mehta and A. Seress, Bounded degree, triangle avoidance graph games, *Graphs Combin.*, to appear.

[13] M.D. Plummer, On n-extendable graphs. *Discrete Math.* 31 (1980), no. 2, 201–210.

[14] P. Prałat, A note on the one-colour avoidance game on graphs, *J. Combin. Math. Combin. Comput.* 75 (2010), 75–85.

[15] A. Seress, On Hajnal's triangle-free game, *Graphs Combin.* 8 (1992), no. 1, 75–79.

[16] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok* 48 (1941), 436–452.

[17] J. Wise and E. Yeager, Game matching numbers of disjoint unions of paths, in preparation.