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NOTE

A Convolution Formula for the Tutte Polynomial

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Let M be a finite matroid with rank function r. We will write $A \subseteq M$ when we mean that A is a subset of the ground set of M, and write $M|_A$ and M/A for the matroids obtained by restricting M to A and contracting M on A respectively. Let M^* denote the dual matroid to M. (See [1] for definitions). The main theorem is

THEOREM 1. *The Tutte polynomial $T_M(x, y)$ satisfies*

$$T_M(x, y) = \sum_{A \subseteq M} T_{M|_A}(0, y)\, T_{M/A}(x, 0). \tag{1}$$

To prove this we first define a convolution product and note a useful lemma. Let $\mathbb{M}$ be the set of all isomorphism classes of finite matroids, and let K be a commutative ring with 1. For any functions $f, g\colon \mathbb{M} \to K$, define $f \circ g\colon \mathbb{M} \to K$ by

$$(f \circ g)(M) = \sum_{A \subseteq M} f(M|_A)\, g(M/A). \tag{2}$$

The convolution $\circ$ is associative, with identity element δ,

$$\delta(M) = \begin{cases} 1 & \text{if} \quad M = \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Following Crapo [2], let $\zeta(x, y)(M) = x^{r(M)} y^{r(M^*)}$, where $K = \mathbb{Z}[x, y]$.

LEMMA 1. $\zeta(x, y)^{-1} = \zeta(-x, -y)$.

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Proof. Note that

$$
\begin{aligned}
&(\zeta(x, y) \circ \zeta(-x, -y))(M) \\
&\quad = \sum_{A \subseteq M} x^{r(M|_A)} y^{r((M|_A)^*)} (-x)^{r(M/A)} (-y)^{r((M/A)^*)} \\
&\quad = x^{r(M)} y^{r(M^*)} \sum_{A \subseteq M} (-1)^{|M| - |A|} \\
&\quad = \delta(M). \quad \blacksquare
\end{aligned}
$$

Proof of Theorem 1. The Tutte polynomial may be defined by [1, 2]

$$T_M(x+1, y+1) = (\zeta(1, y) \circ \zeta(x, 1))(M), \tag{3}$$

so also

$$
\begin{aligned}
T_M(x+1, 0) &= (\zeta(1, -1) \circ \zeta(x, 1))(M), \\
T_M(0, y+1) &= (\zeta(1, y) \circ \zeta(-1, 1))(M).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{A \subseteq M} T_{M|_A}(0, y+1)\, T_{M/A}(x+1, 0) \\
&\quad = (\zeta(1, y) \circ \zeta(-1, 1)) \circ (\zeta(1, -1) \circ \zeta(x, 1))(M) \\
&\quad = \zeta(1, y) \circ (\zeta(-1, 1) \circ \zeta(1, -1)) \circ \zeta(x, 1)(M) \\
&\quad = \zeta(1, y) \circ \zeta(x, 1)(M) \\
&\quad = T_M(x+1, y+1),
\end{aligned}
$$

where the third equality is by Lemma 1. $\blacksquare$

Note that Theorem 1 can be rewritten as

$$T_M(x, y) = \sum_{\text{isthmus-free flats } V} T_V(0, y)\, T_{M/V}(x, 0). \tag{4}$$

This is because when $A \subseteq M$ is not a flat, M/A contains a loop e and

$$T_{M/A}(x, 0) = [\, y T_{(M/A)-e}(x, y)]_{y=0} = 0.$$

Similarly if A contains an isthmus e, then

$$T_{M|_A}(0, y) = [\, x T_{(M|_A)/e}(x, y)]_{x=0} = 0.$$

Theorem 1 also follows immediately from Theorem 5.1 in [3], using Tutte's original definition of $T_M(x, y)$ via *basis activities*. It can also be proven via deletion–contraction.

As an application of Theorem 1, we note that it is closely related to the interpretation of the Tutte polynomial given in [5]. For example, it can be used to give a quick proof of the following:

THEOREM 2 [5, Corollary 2]. *Let G be a graph with $v(G)$ vertices, $e(G)$ edges, and $c(G)$ connected components. Then for any positive integers p, q; its Tutte polynomial T_G satisfies*

$$T_G(1-p, 1-q) = (-p)^{-c(G)} (-1)^{v(G)} \sum_{(x,\, y)} (-1)^{|\mathrm{supp}(\mathbf{y})|},$$

where the sum runs over pairs $(\mathbf{x}, \mathbf{y})$ *in which*

- $\mathbf{x}$ *is a vertex coloring of G with p colors,*
- $\mathbf{y}$ *is a flow on the edges of G with values in any abelian group of cardinality q, and*
- *each edge contains non-zero flow if and only if it is colored improperly.*

Here $|\mathrm{supp}(\mathbf{y})|$ *is the number of edges containing non-zero flow in* $\mathbf{y}$, *or equivalently, the number of improperly colored edges in* $\mathbf{x}$.

Proof. Classify the pairs $(\mathbf{x}, \mathbf{y})$ occurring on the right-hand side of the theorem according to the set $A = \mathrm{supp}(\mathbf{y})$. Then $\mathbf{y}$ is equivalent to a nowhere-zero flow on the edge-subgraph $G|_A$, and $\mathbf{x}$ is equivalent to a proper coloring of the contraction G/A. Now use the usual interpretations of

$$p^{c(G)}(-1)^{v(G)-c(G)}\, T_G(1-p, 0)$$

$$(-1)^{e(G)-v(G)+c(G)}\, T_G(0, 1-q)$$

in terms of the chromatic polynomial counting proper colorings, and the flow polynomial counting nowhere-zero flows [1, Propositions 6.3.1, 6.3.4], respectively. The result then follows from Theorem 1. ∎

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