RECURRENCE RELATIONS FOR THE SPECTRUM POLYNOMIAL OF A MATROID

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Abstract. Combinatorial Laplace operators on the simplicial complex of independent sets of a loopless matroid M are known to have non-negative integral spectra [6]. The spectrum polynomial of M, a polynomial in two variables formulated via the flats of M, is a generating function for the spectra of these operators. In this paper we establish recurrence formulas for the spectrum polynomial of a matroid, analogous to the deletion-contraction recursions for the Tutte polynomial. However, we show that for any matroid M and $e \in M$ the new formulas for the spectrum polynomial depend on whether or not e is closed in M. In particular, the spectrum polynomial is a new invariant for matroids that is not a Tutte-Grothendieck invariant.

1. Introduction

Kook, Reiner, and Stanton showed in [6] that for a matroid M without loops the combinatorial Laplace operators on the independence complex $IN(M)$ have non-negative integral spectra, and that the same information as the spectra is carried by the *spectrum polynimial*

$$\mathrm{Spec}_M(t,q) = \sum_{V,W \in L(M)} |\tilde{\chi}(IN(V))||\mu_{L(M)}(V,W)|t^{r(W)}q^{|V|},$$

where $\tilde{\chi}(IN(V))$ is the reduced Euler characteristic of the independence complex of a flat V and $\mu_{L(M)}$ the Möbius function on the lattice of flats of M. First we recall the definition of the combinatorial Laplace operators for any finite simplicial complex K. Let $\{C_i, \partial_i\}$ $(i \geq -1)$ be the simplicial chain complex of K over the field of real numbers $\mathbb{R}$ with $C_{-1} = \mathbb{R}$. For each i, we have $C_i \cong \mathbb{R}^{d_i}$ where d_i is the number of simplices of dimension i. Regard each boundary map $\partial_i : C_i \to C_{i-i}$ as a $d_{i-1} \times d_i$ matrix. Then we have the (coboundary) map $\partial_i^T : C_{i-1} \to C_i$. The combinatorial Laplacian $\Delta_i : C_i \to C_i$ is given by $\Delta_i = \partial_i^T \partial_i + \partial_{i+1}\partial_{i+1}^T$. One of the main properties of the combinatorial Laplacian is that the 0-eigenspace of Δ_i is naturally isomorphic to $\tilde{H}_i(K;\mathbb{R})$ [4] (refer to [5, Proposition 2.1] for a simple proof), which is an analogue of a fact from Hodge Theory.

For $K = IN(M)$ the independence complex of a matroid M, let Δ_i^M denote the combinatorial Laplacian on $IN(M)$ and $(\Delta_i^M)_\lambda$ the λ-eigenspace of Δ_i^M . Then we have [6, Corollary 17]

$$\sum_{\lambda \in R, i \geq -1} \dim(\Delta_i^M)_\lambda q^\lambda t^i = t^{-1} q^{|M|} \mathrm{Spec}_M(t, q^{-1}),$$

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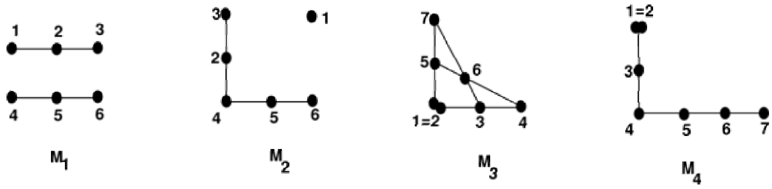

FIGURE 1. Matroids M_1 and M_2 have the same Tutte polynomial and the same spectrum polynomial. Matroids M_3 and M_4 have the same Tutte polynomial but different spectrum polynomials.

which shows that all Laplacians $(\Delta_i^M)_\lambda$ for any matroid M have non-negative integral spectra and that the spectrum polynomial carries the same information as the spectra.

The two matroid invariants involved in Spec_M (short for $\mathrm{Spec}_M(t,q)$), the unsigned reduced Euler characteristic and the Möbius invariant, are (generalized) Tutte-Grothendieck invariants (refer to [2]). Specifically, for any matroid M,

$$|\tilde{\chi}(IN(M))| = T_M(0,1) \quad \text{and} \quad |\mu_{L(M)}(\hat{0}, M)| = T_M(1,0)\,,$$

where $T_M(x,y)$ is the Tutte polynomial of M. Both of these invariants can be computed via deletion and contraction:

$$T_M(x,y) = T_{M-e}(x,y) + T_{M/e}(x,y).$$

when $e \in M$ is not an isthmus nor a loop.

A natural question to ask is whether there is some way to use deletion-contraction to compute the spectrum polynomial Spec_M. First of all, the following examples show that Spec_M is not a (generalized) Tutte-Grothendieck invariant. For example, let M_3 and M_4 be as in Figure 1. Then one can show that M_3 and M_4 have the same Tutte polynomial ([2])

$$y^4 + 3y^3 + 2xy^2 + x^2y + x^3 + 4y^2 + 5xy + 3x^2 + 2x + 2y$$

but different spectrum polynomials ([6])

$$\begin{aligned} \mathrm{Spec}_{M_3}(t,q) &= 1 + t(6+q^2) + t^2(11 + 3q^2 + 2q^3 + 4q^4) + t^3(6 + 2q^2 + 2q^3 + 4q^4 + 10q^7) \\ \mathrm{Spec}_{M_4}(t,q) &= 1 + t(6+q^2) + t^2(11 + 4q^2 + 5q^4) + t^3(6 + 3q^2 + 5q^4 + 10q^7). \end{aligned}$$

Second of all, the examples M_3 and M_4 show that there can be no formula for Spec_M purely as a function of Spec_{M-e} and $\mathrm{Spec}_{M/e}$ because $M_3 - e \cong M_3 - e$ and $M_3/e \cong M_4/e$, where e is the element labeled by 7 in both M_3 and M_4. However we show in this paper that one can recover Spec_M from the knowledge of Spec_{M-e} and $\mathrm{Spec}_{M/e}$ together with the spectrum polynomial defined on the upper intervals $[e, M]$ if e is closed and $[\bar{e} - e, M - e]$ if e is not closed. For example, we will see that the spectrum polynomials on $[e, M_3]$ and $[e, M_4]$ are different although these two intervals are isomorphic.

Throughout this paper we will assume that the matroids considered are without loops. For combinatorial Laplacians are applied to independence complexes of matroids which ignore loops, if any. In section 2 we establish the deletion-contraction formula for $\mathrm{Spec}_M(t,q)$ for the case when $e \in M$ is closed. In section 3 we establish the formula when $e \in M$ is not closed. We refer the reader to the excellent article [1] for almost all of the definitions and notions from matroid theory and matroid complexes that we will use.

2. Deletion-contraction formula for $\mathrm{Spec}_M(t,q)$ Part I

Throughout this section we will assume $e \in M$ is fixed and it is closed in M. First we review some basic facts about the independence complex $IN(M)$ of a matroid M, i.e., the simplicial complex whose i-simplices are the rank $i+1$ independent sets of M. Recall that $IN(M)$ is shellable and the reduced homology $\tilde{H}_i(IN(M))$ is non-zero only in the top dimension $i = r-1$, where r is the rank of M. Therefore the rank of $\tilde{H}_{r-1}(IN(M))$ is an invariant of M, which we will refer to as the α-invariant of M denoted by $\alpha(M)$. We have various descriptions of $\alpha(M)$ as follows:

$$\alpha(M) = \mathrm{rk}\tilde{H}_{r-1}(IN(M)) = |\tilde{\chi}(IN(M))| = T_M(0,1) = \bar{\mu}(M^*),$$

where $\bar{\mu}(M^*)$ is the Möbius invariant of the dual matroid M^*.

Notation: Given a flat $F \in L(M)$ let $\iota(F)$ denote the set of isthmuses in F.

We have $\alpha(F) \neq 0$ iff $\iota(F) = \phi$, i.e. iff F is isthmus free. The flats of M for which the given element $e \in M$ is not an isthmus will play important roles in this section. (The following lemma is valid when $e \in M$ is not necessarily closed in M)

Lemma 1. *Given $e \in M$ let $L(M)_*$ be the sublattice of $L(M)$ consisting of the flats $F \in L(M)$ such that $e \notin \iota(F)$, i.e. e is not an isthmus in F. Then the order-preserving map $f : L(M)_* \to L(M-e)$ given by $f(F) = F - e$ is an isomorphism.*

Proof. f is clearly order-preserving. To show f is one-to-one, suppose $F_1 - e = F_2 - e$ for $F_1, F_2 \in L(M)_*$. Now if e is in only one of F_1 and F_2, say F_1, then $e \notin \iota(F_1)$ implies $F_1 - e$ is not a flat in M whereas $F_2 - e = F_2$ is a flat in M, a contradiction. Hence e must be in both F_1 and F_2 or in neither of them. In either case it follows f is one-to-one. Also it is onto because given $F' \in L(M-e)$ we have $\mathrm{cl}_M(F') - e = F'$ and e cannot be an isthmus in $\mathrm{cl}_M(F')$. □

Let $IL(M)$ denote the set of all intervals $[V,W]$ in $L(M)$. (Similarly for $IL(M-e)$ and $IL(M/e)$.) From the definition of the spectrum polynomial using these notations

$$\mathrm{Spec}_M(t,q) = \sum_{[V,W]\in IL(M)} \alpha(M)|\mu(V,W)|t^{r(W)}q^{|V|}$$

it is clear that non-zero contributions come from $[V,W]$ with $\iota(V) = \phi$. In particular, for a fixed $e \in M$ (again, we assume e is closed in this section.) we may compute $\mathrm{Spec}_M(t,q)$ using only those intervals $[V,W]$ such that $e \notin \iota(V)$. We denote the set of such intervals by $I_*L(M)$. The proof of the deletion-contraction formula for $\mathrm{Spec}_M(t,q)$ will use the following partition of $I_*L(M) = A \sqcup B \sqcup C \sqcup D$, where

$$\begin{aligned} A &= \{[V,W] \in I_*L(M) | e \in V\} \\ B &= \{[V,W] \in I_*L(M) | e \notin V, e \notin \iota(W),\ e \in \iota(F) \text{ for some } F \in (V,W)\} \\ C &= \{[V,W] \in I_*L(M) | e \notin V, e \in \iota(W)\}, \text{and} \\ D &= \{[V,W] \in I_*L(M) | e \notin V, e \notin \iota(F) \text{ for all } F \in [V,W]\}. \end{aligned}$$

Remarks: 1. For $[V,W] \in A$ it is clear that $e \notin \iota(W)$. Otherwise $V \subset W$ would imply $e \in \iota(V)$. In the definition of B, (V,W) denotes an open interval. B consists of $[V,W]$ for which e is an isthmus for an interior flat $F \in (V,W)$ but not for V or W; C consists of $[V,W]$ for which e is an isthmus for W; D consists of the intervals for which e is not an isthmus for any flat in the interval.

2. Note that for $[V,W] \in B \sqcup C$, V is a flat in $M - e$ because $e \notin V$. Also we may regard V as a flat in M/e. Indeed the fact that e is an isthmus for some $F \in (V,W]$ implies that $V \cup e$ is a flat and e is an isthmus in $V \cup e$. Therefore $V = (V \cup e)/e$.

3. As we shall see later, the contributions to $\mathrm{Spec}_M(t,q)$ from $A \sqcup B$ will be captured by $\mathrm{Spec}_{M-e}(t,q)$ and $\mathrm{Spec}_{M/e}(t,q)$, those from C by $\mathrm{Spec}_{M/e}(t,q)$ only, and those from D by $\mathrm{Spec}_{M-e}(t,q)$ only.

Lemma 2. *1. Let $\mathcal{D} = A \cup B \cup D$. The map $d : \mathcal{D} \to IL(M-e)$ given as follows is a bijection:*

$$d([V,W]) = [V-e, W-e]$$

2. Let $\mathcal{C} = A \cup B \cup C$. The map $c : \mathcal{C} \to IL(M/e)$ given as follows is a bijection:

$$c([V,W]) = \begin{cases} [V/e, W/e] & \textit{if} \quad [V,W] \in A \\ [V, W/e] & \textit{if} \quad [V,W] \in B \cup C. \end{cases}$$

Proof of 1. From the definitions of A, B, and D it is clear that $\mathcal{D} = A \cup B \cup D$ consists of intervals $[V,W] \in L(M)$ such that $e \notin \iota(V)$ and $e \notin \iota(W)$. The result follows from lemma 1.

Proof of 2. Note that the map c makes sense by the above remark 2. Recall that $[e,M]$ is isomorphic to $L(M/e)$ via the contraction $F \mapsto F/e$. Then for each $[F,W] \subset [e,M]$ the map given by $[F,W] \mapsto [F/e, W/e]$ on the intervals is an isomorphism from $I[e,M]$ the set of all intervals contained in $[e,M]$ to $IL(M/e)$. If $e \notin \iota(F)$ then $[F,W]$ is in A and this map is the map c as above. If $e \in \iota(F)$, then we may factor the map $[F,W] \mapsto [F/e, W/e]$ through $I_*L(M)$ as follows: $[F,W] \mapsto [F/e, W] \overset{c}{\mapsto} [F/e, W/e]$, where it is clear that $[F/e, W] \in B \cup C$. It also follows from the definitions of B and C that the map $[F,W] \mapsto [F/e, W]$ is a bijection from the set of intervals in $[e,M]$ with $e \in \iota(F)$ onto $B \cup C$. Hence the lemma. □

Notation: Since $\mathcal{D} = A \cup B \cup D$ and $\mathcal{C} = A \cup B \cup C$ map bijectively onto $IL(M-e)$ and $IL(M/e)$ under the maps d and c, respectively, we will denote the corresponding partitions of these sets as follows:

$$IL(M-e) = A_d \sqcup B_d \sqcup D_d \quad \text{and} \quad IL(M/e) = A_c \sqcup B_c \sqcup C_c.$$

In the following lemma we describe the relations among the values of the Möbius functions on the elements in $I_*L(M)$, $I(M-e)$, and $I(M/e)$. Again we assume $e \in M$ is closed in M (loopless).

Lemma 3. *(a) Let $[V,W] \in A$, i.e. $e \in V$, $e \notin \iota(V)$, and $e \notin \iota(W)$. Then $[V,W]$ is isomorphic, as a lattice, to its images $[V-e, W-e] \in A_d$ and $[V/e, W/e] \in A_c$. In particular we have*

$$\mu_M(V,W) = \mu_{M-e}(V-e, W-e) = \mu_{M/e}(V/e, W/e).$$

(b) Let $[V,W] \in B$, i.e. $e \notin V$, $e \notin \iota(W)$, and $e \in \iota(F)$ for some $F \in (V,W)$. For its images $[V, W-e] \in B_d$ and $[V, W/e] \in B_c$ we have

$$\mu_M(V,W) = \mu_{M-e}(V, W-e) + \mu_{M/e}(V, W/e).$$

(c) Let $[V, W] \in C$, i.e. $e \notin V$ and $e \in \iota(W)$. For its image $[V, W/e] \in C_c$ we have

$$\mu_M(V, W) = \mu_{M/e}(V/e, W/e).$$

(d) Let $[V, W] \in D$, i.e. $e \notin V$ and $e \notin \iota(F)$ for all $F \in [V, W]$. For its image $[V, W - e] \in D_d$ we have $\mu_M(V, W) = \mu_{M-e}(V, W - e)$.

Proof. (a) Clearly $[V, W]$ is isomorphic to $[V/e, W/e]$. We claim that the map $f : [V, W] \to [V - e, W - e]$ given by $F \mapsto F - e$ is a lattice isomorphism. f is clearly one-to-one and order preserving. To show f is onto note that $e \notin \iota(V)$ and $e \notin \iota(W)$ imply $\mathrm{cl}_M(V - e) = V$ and $\mathrm{cl}_M(W - e) = W$, respectively. Therefore given any F' with $V - e \leq F' \leq W - e$, we must have $V \leq \mathrm{cl}_M(F) \leq W$. Since $F' = \mathrm{cl}_M(F') - e$, f is onto.
(b) Since we have $e \notin V$ and $e \notin \iota(W)$, e is not an isthmus nor a loop in W/V. Also $e \in \iota(F)$ for some $F \geq V$ implies $V \cup e$ is a flat. Therefore $(W/V)/e = W/(V \cup e) = (W/e)/V$ is loopless. By the deletion-contraction formula for the Möbius invariant we have

$$\begin{aligned} |\mu_M(V, W)| &= |\mu(\hat{0}, W/V)| \\ &= |\mu(\hat{0}, (W - e)/V)| + |\mu(\hat{0}, W/(V \cup e))| \\ &= |\mu_{M-e}(V, W - e)| + |\mu_{M/e}(V, W/e)|. \end{aligned}$$

(c) $e \in \iota(W)$ implies $[V, W]$ is isomorphic to $[V, W/e] \times [\hat{0}, e]$.
(d) The map $g : [V, W] \to [V, W - e]$ given by $F \mapsto F - e$ is a lattice isomorphism by a similar argument as in (a). □

For any subset $X \subset IL(M)$, define

$$\mathrm{Spec}_X(t, q) = \sum_{[F_1, F_2] \in X} \alpha(F_1)|\mu(F_1, F_2)| t^{r(F_2)} q^{|F_1|}.$$

With this definition we have (Spec_M short for $\mathrm{Spec}_M(t, q)$ and similarly for others.)

$$\begin{aligned} \mathrm{Spec}_M &= \mathrm{Spec}_{I_*L(M)} &= \mathrm{Spec}_A + \mathrm{Spec}_B + \mathrm{Spec}_C + \mathrm{Spec}_D \\ \mathrm{Spec}_{M-e} &= \mathrm{Spec}_{IL(M-e)} &= \mathrm{Spec}_{A_d} + \mathrm{Spec}_{B_d} + \mathrm{Spec}_{D_d} \\ \mathrm{Spec}_{M/e} &= \mathrm{Spec}_{IL(M/e)} &= \mathrm{Spec}_{A_c} + \mathrm{Spec}_{B_c} + \mathrm{Spec}_{C_c} \end{aligned}$$

We will also use the notation $I_*[e, M]$ for the set $A = \{[V, W] \in I_*L(M) | e \in V\}$. Again it is the set of all intervals $[V, W]$ contained in the upper interval $[e, M]$ of $L(M)$ such that e is not an isthmus in V (hence not an isthmus for any $F \in [V, W]$). Now we are ready to derive the deletion-contraction formula for $\mathrm{Spec}_M(t, q)$ when $e \in M$ is closed.

Theorem 4. *For a loopless matroid M, let $e \in M$ be closed. Then we have*

$$\mathrm{Spec}_M(t, q) = \mathrm{Spec}_{M-e}(t, q) + t\mathrm{Spec}_{M/e}(t, q) + (1 - q^{-1})\mathrm{Spec}_{I_*[e,M]}(t, q).$$

Proof. Given $[V, W] \in A$, e is not an isthmus nor a loop in V. Therefore we have $\alpha(V) = \alpha(V - e) + \alpha(V/e)$. Also $e \notin \iota(W)$ implies $r(W) = r(W - e)$. Now by the lemma 3(a) we have

$$\begin{aligned} \alpha(V)|\mu(V, W)| t^{r(W)} q^{|V|} &= (\alpha(V - e) + \alpha(V/e))|\mu(V, W)| t^{r(W)} q^{|V|} \\ &= \alpha(V - e)|\mu(V - e, W - e)| t^{r(W-e)} q^{|V-e|+1} \\ &\quad + \alpha(V/e)|\mu(V/e, W/e)| t^{r(W/e)+1} q^{|V/e|+1}, \end{aligned}$$

from which it follows that $\text{Spec}_A = q(\text{Spec}_{A_d} + t\cdot\text{Spec}_{A_c})$. Given $[V,W]\in B$ we also have $e\notin \iota(W)$. Therefore we have by the lemma 3(b)

$$\begin{aligned}\alpha(V)|\mu(V,W)|t^{r(W)}q^{|V|} &= \alpha(V)(|\mu(V,W-e)|+|\mu(V,W/e)|)t^{r(W)}q^{|V|}\\ &= \alpha(V)|\mu(V,W-e)|t^{r(W-e)}q^{|V|}\\ &\quad+\alpha(V)|\mu(V,W/e)|t^{r(W/e)+1}q^{|V|},\end{aligned}$$

from which it follows that $\text{Spec}_B = \text{Spec}_{B_d} + t\cdot\text{Spec}_{B_c}$. From the lemma 3(c) and 3(d) it also follows by similar computations as above that $\text{Spec}_C = t\cdot\text{Spec}_{C_c}$ and $\text{Spec}_D = \text{Spec}_{D_d}$. Finally we have

$$\begin{aligned}&\text{Spec}_M - (1-q^{-1})\text{Spec}_A\\ =\ &\text{Spec}_A + \text{Spec}_B + \text{Spec}_C + \text{Spec}_D - (1-q^{-1})\text{Spec}_A\\ =\ &(\text{Spec}_{A_d} + t\cdot\text{Spec}_{A_c}) + (\text{Spec}_{B_d} + t\cdot\text{Spec}_{B_c}) + t\cdot\text{Spec}_{C_c} + \text{Spec}_{D_d}\\ =\ &\text{Spec}_{IL(M-e)} + t\cdot\text{Spec}_{IL(M/e)}\\ =\ &\text{Spec}_{M-e} + t\cdot\text{Spec}_{M/e}.\end{aligned}$$

($I_*[e,M]$ is another notation for A.) Hence the theorem. □

Example. Let M_3 and M_4 be as in Figure 1. Then

$$\begin{aligned}\text{Spec}_{I_*[e,M_3]}(t,q) &= t^2(q^3+2q^4)+t^3(q^3+2q^4+10q^7)\\ \text{Spec}_{I_*[e,M_4]}(t,q) &= t^2(3q^4)+t^3(3q^4+10q^7)\end{aligned}$$

The difference in $\text{Spec}_{I_*[e,M_3]}(t,q)$ and $\text{Spec}_{I_*[e,M_4]}(t,q)$ explains why M_3 and M_4 have different spectrum polynomials even when $M_3-e\cong M_4-e$ and $M_3/e\cong M_4/e$.

Remark: If e is an isthmus in M, it is an isthmus for every flat containing e. Therefore $I_*[e,M]$ is empty and $M-e\cong M/e$, and the above formula reduces to $\text{Spec}_M(t,q) = (1+t)\text{Spec}_{M/e}(t,q)$. Since we have $\text{Spec}_e(t,q) = (1+t)$, it is consistent with the property ([6]) $\text{Spec}_{M_1\oplus M_2}(t,q) = \text{Spec}_{M_1}(t,q)\cdot\text{Spec}_{M_2}(t,q)$.

3. Deletion-contraction formula for $\text{Spec}_M(t,q)$ Part II

In this section we prove the deletion-contraction formula for $\text{Spec}_M(t,q)$ when an element $e\in M$ that is not closed. We will denote the closure of e in M by $\bar{e}$. Let $I[\bar{e}-e, M-e]$ denote the subset of $IL(M-e)$ consisting of all intervals $[V,W]$ contained in $[\bar{e}-e, M-e]$.

Theorem 5. *Let M be a loopless matroid and $e\in M$ an element in the ground set that is not closed, i.e. $\bar{e}\neq e$. Then we have*

$$\text{Spec}_M(t,q) = \text{Spec}_{M-e}(t,q) + tq^{|\bar{e}|}\text{Spec}_{M/\bar{e}}(t,q) + (q-1)\text{Spec}_{I[\bar{e}-e,M-e]}(t,q).$$

Proof. In this case e will not appear as an isthmus for any flat in M and $L(M)$ is isomorphic to $L(M-e)$. Consequently every interval $[V,W]\subset L(M)$ is isomorphic to its image $[V-e,W-e]\in L(M-e)$. Now consider the following partition $IL(M) = E\sqcup E'$, where E consists of intervals $[V,W]$ with $e\in V$ and E' with $e\notin V$. We have the corresponding partition $IL(M-e) = E_d\sqcup E'_d$ via the isomorphism from $IL(M)$ to $IL(M-e)$ given by $[V,W]\mapsto[V-e,W-e]$. Also E is isomorphic to $IL(M/e)$

Now given $[V,W]\in E$, we have $[V,W]\cong[V-e,W-e]\cong[V/\bar{e},W/\bar{e}]$, and $r(W) = r(W-e) = r(W/\bar{e})+1$. Moreover e is not an isthmus nor a loop in V

because e is not closed. Hence we have $\alpha(V) = \alpha(V-e) + \alpha(V/e)$. Therefore

$$\begin{aligned} \alpha(V)|\mu(V,W)|t^{r(W)}q^{|V|} &= \alpha(V)|\mu(V-e,W-e)|t^{r(W-e)}q^{|V-e|+1} \\ &\quad +\alpha(V/\bar{e})|\mu(V/\bar{e},W/\bar{e})|t^{r(W/\bar{e})+1}q^{|V/\bar{e}|+|\bar{e}|}, \end{aligned}$$

which implies $\mathrm{Spec}_E = q \cdot \mathrm{Spec}_{E_d} + tq^{|\bar{e}|} \cdot \mathrm{Spec}_{IL(M/e)}$. It is straight forward to check that $\mathrm{Spec}_{E'} = \mathrm{Spec}_{E'_d}$. Therefore

$$\begin{aligned} \mathrm{Spec}_M - (q-1)\mathrm{Spec}_{E_d} &= \mathrm{Spec}_E + \mathrm{Spec}_{E'} - (q-1)\mathrm{Spec}_{E_d} \\ &= (\mathrm{Spec}_{E_d} + tq^{|\bar{e}|}\mathrm{Spec}_{IL(M/e)}) + \mathrm{Spec}_{E'_d} \\ &= \mathrm{Spec}_{IL(M-e)} + tq^{|\bar{e}|}\mathrm{Spec}_{IL(M/e)} \\ &= \mathrm{Spec}_{M-e} + tq^{|\bar{e}|}\mathrm{Spec}_{M/e}. \end{aligned}$$

Clearly $E_d = I[\bar{e}-e, M-e]$. Hence the theorem. □

Extending the work of [6], G. Denham studied the combinatorial Laplacians of the Tutte complex, a double complex graded both by corank and nullity [3]. The eigenvalues are enumerated by the more general spectrum polynomial, a weighted version of a Tutte polynomial, which specializes to both $\mathrm{Spec}_M(t,q)$ and the Tutte polynomial. This suggests the following natural question:

Question 6. *Are there recurrence relations for Denham's spectrum polynomial that will specialize to Theorem 4 and 5 for* $\mathrm{Spec}_M(t,q)$ *and the deletion-contraction recurrence of the Tutte polynomial?*

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