INTRODUCTION TO GRAPH THEORY - GAMES ON FINITE GRAPHS

INTRODUCTION

Definition 1. A path is a collection of ‘vertices’ (dots), and ‘edges’ (lines connecting the dots), so that each vertex (aside from the first has a unique predecessor.

Note 1. $P_n$ denotes the path on $n$ vertices, with $n-1$ edges

Definition 2. A cycle can be thought of as a ‘closed path’. That is, a path with the two endpoints connected by an edge.

Groups I and II

Definition 3. A coloring of a graph is an assignment of colors to each of the vertices of the graph.

Definition 4. A $k$-coloring of a graph is an assignment of colors to each of the vertices of the graph which uses only $k$ colors.

Definition 5. A coloring of a graph is said to be ‘proper’ if no two vertices joined by an edge have the same color.

Game 1. (for Group 1) This game is called Achievement. It can be played on paths or cycles. The game is played by two players who alternate turns choosing a vertex to color. Only one color is available (call it blue), and all of the vertices begin the game being colored white. The goal of the game is to be the last player who is able to assign a color to a vertex without making an ‘improper’ coloring (i.e. to force the other player to improperly color a vertex). Try examples for paths.
of length 3, 5, 7, and 9. Can you make a conjecture about the strategy Player 1 should choose? Now try examples for cycles of length 4, 6, 8, and 10. Again, can you make a conjecture about a strategy for Player 1? Player 2? (ie. if Player 1 plays a certain strategy can he or she guarantee to always win?) How are these two games related?

**Game 2.** (for Group 2) This game is called Avoidance. It can be played on paths or cycles. The game is played by two players who alternate turns choosing a vertex to color. Only one color is available (call it blue), and all of the vertices begin the game being colored white. The goal of the game is to force the opposite player to be the last who is able to assign a color to a vertex without making an ‘improper’ coloring (ie. to leave yourself with no more ‘moves’). Try examples for paths of length 4, 6, 8, and 10. Can you make a conjecture about the strategy Player 1 should choose? Now try examples for cycles of length 3, 5, 7, and 9. Again, can you make a conjecture about a strategy for Player 1? Player 2? (ie. if Player 1 plays a certain strategy can he or she guarantee to always win?) How are these two games related?

**Group III and IV**

A group of Revolutionaries are trying to meet up to discuss a plan for attacking the British Forces during the Revolutionary War. The British Forces have Spies who are available to gain intelligence about the Revolutionaries’ plans by watching over the Revolutionaries closely. If a group of $k$ Revolutionaries are able to congregate at a meeting place (vertex) with no British Spy, then the Revolutionaries will be able to plan a surprise attack on the British Forces and win a battle. The goal of the British Spies is to ensure that this does not happen. Assume that the Spies are placed after the Revolutionaries choose their starting places (so that the Revolutionaries cannot win by merely choosing the correct starting configuration). Consider the following situations:

**Game 3.** Consider $P_n$, where each vertex is an available meeting place for the Revolutionaries. If there are $r$ Revolutionaries and a meeting between $k$ Revolutionaries is required to plan an attack, how many Spies must the British Forces send to ensure that the Revolutionaries cannot have a secret meeting between at least $k$ people? Consider some small cases. Begin with a meeting size of $k = 3$ required, and $r = 5$ Revolutionaries on $P_{10}$. Is 2 Spies enough? 3 Spies? What if we increase the total number of Revolutionaries to $r = 8$? How many Spies are needed? What strategy do the Spies need to use? Can you make a conjecture
about the number of Spies required in the general case where the number of Revolutionaries is \( r \) and the required meeting size is \( k \)? Does the length of the path (number of possible meeting places) matter?

**Game 4.** Consider \( C_n \), where each vertex is an available meeting place for the Revolutionaries. If there are \( r \) Revolutionaries and a meeting between \( k \) Revolutionaries is required to plan an attack, how many Spies must the British Forces send to ensure that the Revolutionaries cannot have a secret meeting between at least \( k \) people? Consider some small cases. Begin with a meeting size of \( k = 3 \) required, and \( r = 5 \) Revolutionaries on \( C_{10} \). Is 2 Spies enough? 3 Spies? What if we increase the total number of Revolutionaries to \( r = 8 \)? How many Spies are needed? What strategy do the Spies need to use? Can you make a conjecture about the number of Spies required in the general case where the number of Revolutionaries is \( r \) and the required meeting size is \( k \)? Does the length of the cycle (number of possible meeting places) matter?

**Game 5.** Consider \( S_{(n,k)} \) (the sun graph shown below), where each vertex is an available meeting place for the Revolutionaries. Note that \( S_{(n,k)} \) consists of a \( C_n \) with paths of length \( k \) attached at each vertex of \( C_n \). That is, the sun graph shown below is \( S_{(5,4)} \)

![Sun Graph](image)

If there are \( r \) Revolutionaries and a meeting between \( k \) Revolutionaries is required to plan an attack, how many Spies must the British Forces send to ensure that the Revolutionaries cannot have a secret meeting between at least \( k \) people? Consider some small cases. Begin with a meeting size of \( k = 3 \) required, and \( r = 5 \) Revolutionaries on \( S_{(10,4)} \). Is 2 Spies enough? 3 Spies? 6 Spies? What if we increase the total number of Revolutionaries to \( r = 8 \)? How many Spies are needed? What strategy do the Spies need to use? Can you make a conjecture about the number of Spies required in the general case where the number of Revolutionaries is \( r \) and the required meeting size is \( k \)? Does the length of the path (number of possible meeting places) matter?

If you are interested in finding out more about this ‘game’, you might search MathSciNet for Douglas West or Spies and Revolutionaries.
Another Type of Game

Consider that there are a group of adjacent stores in a Mall, and all of the stores have Security Cameras. Also, consider that an intruder is loose in the mall. The security cameras are fairly advanced and will send a signal to the security office if the intruder is in the room which they are monitoring, or in any of the rooms which immediately neighbor it (any of the connected stores). However, assume that the intruder is also fairly smart, and has the ability to configure one of the cameras to lie. There are two ways that the camera can lie. It can falsely report that the intruder is currently nearby, or it can neglect to report that the intruder is nearby. The idea here is to minimize the number of stores (or rooms) which need to be monitored by a security camera, but we are still always able to find the intruder. This type of game is called ‘The Liar’s Domination’. Consider the following configuration of adjacent stores. Where should we place security cameras? What is the minimum number of cameras we need in order to be certain that we can find the intruder?

If you are interested in finding out more about this ‘game’, you might search MathSciNet for Peter Slater or Miranda Bowie (Roden) or Liar’s Domination