Area and Arc Length in Polar Coordinates

The Cartesian Coordinate System (rectangular coordinates) is not always the most convenient way to describe points, or relations in the plane. There are certainly other ways to describe where a point lies.

In the Cartesian Coordinate System a point, $P$ is identified by an $x$-coordinate (a horizontal distance from the origin), and a $y$-coordinate (a vertical distance from the origin).

In Polar Coordinates, a point $P$ is identified by its distance from the origin, $r$ (think of this as some radius), and its angle $\theta$, measured counterclockwise from the positive axis. Then $(r, \theta)$ is the location of the point $P$, in the plane, using polar coordinates.

Using simple, right-triangle trigonometry, one can find the relationship between Rectangular and Polar coordinates:

\[
x = r \cos(\theta) \\
y = r \sin(\theta) \\
r = \sqrt{x^2 + y^2} \\
\tan(\theta) = \frac{y}{x}
\]

Note that when converting from rectangular coordinates to polar coordinates, you should plot the point in rectangular coordinates first. This way you can determine which quadrant the point lies in since $\arctan(x)$ only has range $[-\pi/2, \pi/2]$, you can’t recover the quadrant.

**Example 1** Perform the following conversions

a) Give the cartesian coordinates for the point with polar coordinates given by $P = (7, \pi/3)$

b) Give the polar coordinates for the point with cartesian coordinates given by $Q = (3, 4)$
Solution.

a) For this point, $r = 7$ and $\theta = \pi/3$.

\[
x = r \cos(\theta) \\
\implies x = 7 \cos(\pi/3) \\
\implies x = 7/2
\]

\[
y = r \sin(\theta) \\
\implies y = 7 \sin(\pi/3) \\
\implies y = 7\sqrt{3}/2
\]

b) For this point, $x = 3$ and $y = 4$

\[
r = \sqrt{3^2 + 4^2} = 5
\]

Now, the point $(3, 4)$ lies in the first quadrant.

\[
\tan(\theta) = \frac{4}{3}
\]

\[
\implies \theta = \arctan\left(\frac{4}{3}\right) \approx .927 \text{ rad} \quad (57^\circ)
\]

Note that since $\theta$ can wrap around the origin many times, each point in the plane has infinitely many descriptions in polar coordinates.

Example 2  Plot the points $(5, \frac{3\pi}{4})$, $(5, \frac{11\pi}{4})$, and $(5, -\frac{5\pi}{4})$.

Note that we can also plot points with a negative $r$ value. These points are plotted in the opposite direction of the angle $\theta$, that is, in the direction of $\theta + \pi$.

Example 3  Plot the point $(-4, \frac{4\pi}{3})$.

You will not be required to graph a function in polar coordinates on an exam, but you may be required to calculate areas or arc-lengths for functions in polar form.
Functions in Polar Coordinates

Some functions are easier to describe in Polar Coordinates than with rectangular coordinates. See the following example:

**Example 4** Plot the functions $y = 1$ and $r = 1$. Then rewrite the function $y = 1$ in Polar Coordinates, and rewrite the function $r = 1$ in rectangular coordinates.

**Solution.** Note that the equation $y = 1$ is a horizontal line going through the point $(0, 1)$, and the equation $r = 1$ is a circle of radius 1, centered at the origin.

Now, rewriting the equation $y = 1$

$$y = r \sin(\theta)$$

$$\implies 1 = r \sin(\theta)$$

$$\implies r = \frac{1}{\sin(\theta)}$$

The equation $y = 1$ is more simple.

Rewriting the equation $r = 1$

$$r = \sqrt{x^2 + y^2}$$

$$\implies 1 = \sqrt{x^2 + y^2}$$

$$\implies x^2 + y^2 = 1$$

As expected, we obtained the familiar equation for a circle of radius 1 centered at the origin. However, the equation was much more simple to express in terms of polar coordinates.
Area in Polar Coordinates

When calculating area in polar coordinates, we take small intervals of $\theta$ and calculate the area of the corresponding circular sector.

The area of a circular sector with interior angle $\theta$ is given by:

$$A = \frac{\theta}{2\pi} \cdot \pi r^2$$

$$= \frac{1}{2} r^2 \theta$$

Adding these areas together gives an approximate area for the entire region. As we did with rectangular coordinates, we take small increments of $\theta$, which we denote by $\Delta \theta$. Now, our circular sectors have area:

$$A = \frac{1}{2} r^2 \Delta \theta$$

Integrating, gives an exact value for the area.

In general, the area of the region bounded by a curve $r(\theta)$ for $\alpha \leq \theta \leq \beta$ is given by

$$\frac{1}{2} \int_{\alpha}^{\beta} r(\theta)^2 d\theta$$

**Example 5** Find the area of the region bounded by the function $r(\theta) = 3 + 2 \cos(\theta)$ for $0 \leq \theta \leq 2\pi$.

**Solution.**
Example 6 Find the area of one petal of the four-petal rose $r = 3\sin(2\theta)$

Solution.

The petal in the first quadrant is described by $r = 3\sin(2\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$.

$$A = \frac{1}{2} \int_{0}^{\pi/2} (3\sin(2\theta))^2 \, d\theta$$
Example 7 Find the area lying outside $r = \theta$ and inside $r = 2\theta$

Solution.

Our area is given by:

$$\frac{1}{2} \int_{0}^{2\pi} (2\theta)^2 - (\theta)^2 d\theta$$

$$= \frac{1}{2} \int_{0}^{2\pi} 3\theta^2 d\theta$$

$$= \frac{1}{2}(\theta^3|_{0}^{2\pi})$$

$$= 4\pi^3$$

Arc Length in Polar Coordinates

In order to find the arc length of the curve $r = f(\theta)$, we first must express $x$ and $y$ in terms of $\theta$.

$$x = f(\theta) \cos(\theta)$$
$$y = f(\theta) \sin(\theta)$$

The arc length of the curve $r = f(\theta)$ is given by:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$
Example 8  Find the arc length of two petals of the rose \( r = 3 \sin(2\theta) \) for \( 0 \leq \theta \leq \pi \).

Solution.

\[
\begin{align*}
x &= 3 \sin(2\theta) \cos(\theta) \\
\frac{dx}{d\theta} &= -3 \sin(2\theta) \sin(\theta) + 6 \cos(2\theta) \cos(\theta) \\
y &= 3 \sin(2\theta) \sin(\theta) \\
\frac{dy}{d\theta} &= 3 \sin(2\theta) \cos(\theta) + 6 \cos(2\theta) \sin(\theta)
\end{align*}
\]

So, the arc length is given by:

\[
L = \int_{0}^{\pi} \sqrt{(-3 \sin(2\theta) \sin(\theta) + 6 \cos(2\theta) \cos(\theta))^2 + (3 \sin(2\theta) \cos(\theta) + 6 \cos(2\theta) \sin(\theta))^2} \, d\theta
\]