Global Behavior of Two Competitive Rational Systems of Difference Equations in the Plane

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Abstract

We investigate the global dynamics of solutions of two distinct competitive rational systems of difference equations in the plane. We show that the basins of attraction of different locally asymptotically stable equilibrium points are separated by the global stable manifolds of either saddle points or of non-hyperbolic equilibrium points. Our results give complete answer to Open Problem 1 posed recently in [2].

Key Words: asymptotic behavior, global, manifolds, monotone maps, stability

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1 Introduction and Preliminaries

We consider the following open problem (see Open Problem 1 in [2]):

For each of the following two distinct systems

\[ x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{B_2 x_n + y_n}, \quad n = 0, 1, \ldots, \quad (1) \]

and

\[ x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}, \quad n = 0, 1, \ldots, \quad (2) \]

determine the following:

(i) The boundedness character of its solutions;

(ii) The local stability of its equilibrium points;

(iii) The global character of the systems.

In the classification from [2]) the first system is known as (14, 15) and the second system is known as (14, 38).

The solution to part (i) for both systems (1) and (2) is the following result

**Theorem 1** Every solution \( \{(x_n, y_n)\} \) of systems (1) and (2) is bounded if and only if \( \beta_1 \leq A_1 \). More precisely, \( \{x_n\} \) is bounded if and only if \( \beta_1 \leq A_1 \) and \( \{y_n\} \) is always bounded.

The answers to parts (ii) and (iii) are given in Sections 2 and 3.

Systems (1) and (2) are competitive, which we discuss next.

A first order system of difference equations

\[
\begin{cases}
  x_{n+1} = f(x_n, y_n) \\
  y_{n+1} = g(x_n, y_n)
\end{cases}, \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in \mathcal{R},
\]

where \( \mathcal{R} \subset \mathbb{R}^2 \), \( (f, g) : \mathcal{R} \to \mathcal{R} \), \( f \), \( g \) are continuous functions is competitive if \( f(x, y) \) is non-decreasing in \( x \) and non-increasing in \( y \), and \( g(x, y) \) is non-increasing in \( x \) and non-decreasing in \( y \). If both \( f \) and \( g \) are nondecreasing in \( x \) and \( y \), the system (3) is cooperative. Competitive and cooperative maps are defined similarly. Strongly competitive systems of difference equations or strongly competitive maps are those for which the functions \( f \) and \( g \) are coordinate-wise strictly monotone.

If \( \mathbf{v} = (u, v) \in \mathbb{R}^2 \), we denote with \( Q_\ell(\mathbf{v}) \), \( \ell \in \{1, 2, 3, 4\} \), the four quadrants in \( \mathbb{R}^2 \) relative to \( \mathbf{v} \), i.e., \( Q_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u, \ y \geq v\} \), \( Q_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u, \ y \geq v\} \), and so on. Define the South-East partial order \( \preceq_{\text{se}} \) on \( \mathbb{R}^2 \) by \( (x, y) \preceq_{\text{se}} (s, t) \) if and only if \( x \leq s \) and \( y \geq t \). Similarly, we define the North-East partial order \( \preceq_{\text{ne}} \) on \( \mathbb{R}^2 \) by \( (x, y) \preceq_{\text{ne}} (s, t) \) if and only if \( x \leq s \) and \( y \leq t \). For \( \mathcal{A} \subset \mathbb{R}^2 \) and \( x \in \mathbb{R}^2 \), define the distance from \( x \) to \( \mathcal{A} \) as \( \text{dist}(x, \mathcal{A}) := \inf \{\|x - y\| : y \in \mathcal{A}\} \). By \( \text{int} \mathcal{A} \) we denote the interior of a set \( \mathcal{A} \).

It is easy to show that a map \( F \) is competitive if it is non-decreasing with respect to the South-East partial order, that is if the following holds:
For standard definitions of attracting fixed point, saddle point, stable manifold, and related notions see [10] and [17].

Competitive systems were studied by many authors see [1, 3, 4, 6, 7, 8, 10, 14, 15, 16, 18, 19, 20, 21, 22, 23], and others. All known results, with the exception of [3, 13], deal with hyperbolic dynamics. The results from the literature presented next are results that hold in both the hyperbolic and the non-hyperbolic case.

We now state three results for competitive maps in the plane. The following definition is from [23].

**Definition 1** Let \( R \) be a nonempty subset of \( \mathbb{R}^2 \). A competitive map \( T : R \to R \) is said to satisfy condition \((O+)\) if for every \( x, y \in R \), \( T(x) \preceq_{ne} T(y) \) implies \( x \preceq_{ne} y \), and \( T \) is said to satisfy condition \((O−)\) if for every \( x, y \in R \), \( T(x) \preceq_{ne} T(y) \) implies \( y \preceq_{ne} x \).

The following theorem was proved by DeMottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [20, 21].

**Theorem 2** Let \( R \) be a nonempty subset of \( \mathbb{R}^2 \). If \( T \) is a competitive map for which \((O+)\) holds then for all \( x \in R \), \( \{T^n(x)\} \) is eventually componentwise monotone. If the orbit of \( x \) has compact closure, then it converges to a fixed point of \( T \). If instead \((O−)\) holds, then for all \( x \in R \), \( \{T^{2n}\} \) is eventually componentwise monotone. If the orbit of \( x \) has compact closure in \( R \), then its omega limit set is either a period-two orbit or a fixed point.

The following result is from [23], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions \((O+)\) and \((O−)\).

**Theorem 3** Let \( R \subset \mathbb{R}^2 \) be the cartesian product of two intervals in \( \mathbb{R} \). Let \( T : R \to R \) be a \( C' \) competitive map. If \( T \) is injective and \( \det J_T(x) > 0 \) for all \( x \in R \) then \( T \) satisfies \((O+)\). If \( T \) is injective and \( \det J_T(x) < 0 \) for all \( x \in R \) then \( T \) satisfies \((O−)\).

The next result is a modification of Theorem 4 from [11]. See also [12].

**Theorem 4** Let \( T \) be a monotone map on a closed and bounded rectangular region \( \mathcal{R} \subset \mathbb{R}^2 \). Suppose that \( T \) has a unique fixed point \( \bar{e} \) in \( \mathcal{R} \). Then \( \bar{e} \) is a global attractor of \( T \) on \( \mathcal{R} \).

The following four results were proved by Kulenović and Merino [13] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. These results are useful for determining basins of attraction of fixed points of competitive maps.

Our first result gives conditions for the existence of a global invariant curve through a fixed point (hyperbolic or not) of a competitive map that is differentiable in a neighborhood of the fixed point, when at least one of two nonzero eigenvalues of the Jacobian matrix of the map at the fixed point has absolute value less than one. A region \( \mathcal{R} \subset \mathbb{R}^2 \) is rectangular if it is the cartesian product of two intervals in \( \mathbb{R} \).
**Theorem 5** Let $T$ be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\mathbf{x} \in \mathcal{R}$ be a fixed point of $T$ such that $\Delta := \mathcal{R} \cap \text{int}(Q_1(\mathbf{x}) \cup Q_3(\mathbf{x}))$ is nonempty (i.e., $\mathbf{x}$ is not the NW or SE vertex of $\mathcal{R}$), and $T$ is strongly competitive on $\Delta$. Suppose that the following statements are true.

a. The map $T$ has a $C^1$ extension to a neighborhood of $\mathbf{x}$.

b. The Jacobian matrix of $T$ at $\mathbf{x}$ has real eigenvalues $\lambda, \mu$ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace $E^\lambda$ associated with $\lambda$ is not a coordinate axis.

Then there exists a curve $C \subset \mathcal{R}$ through $\mathbf{x}$ that is invariant and a subset of the basin of attraction of $\mathbf{x}$, such that $C$ is tangential to the eigenspace $E^\lambda$ at $\mathbf{x}$, and $C$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $C$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $C$ is a minimal period-two orbit of $T$.

**Corollary 1** If $T$ has no fixed point nor periodic points of minimal period two in $\Delta$, then the endpoints of $C$ belong to $\partial \mathcal{R}$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 5 reduces just to $|\lambda| < 1$. This follows from a change of variables [23] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis.

The following result gives a description of the global stable and unstable manifolds of a saddle point of a competitive map. The result is the modification of Theorem 5 from [11]. See also [12].

**Theorem 6** In addition to the hypotheses of Theorem 5, suppose that $\mu > 1$ and that the eigenspace $E^\mu$ associated with $\mu$ is not a coordinate axis. If the curve $C$ of Theorem 5 has endpoints in $\partial \mathcal{R}$, then $C$ is the global stable manifold $W^s(\mathbf{x})$ of $\mathbf{x}$, and the global unstable manifold $W^u(\mathbf{x})$ is a curve in $\mathcal{R}$ that is tangential to $E^\mu$ at $\mathbf{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $W^u(\mathbf{x})$ in $\mathcal{R}$ are fixed points of $T$.

The next result from [13] is useful for determining basins of attraction of fixed points of competitive maps.

**Theorem 7** Assume the hypotheses of Theorem 5, and let $C$ be the curve whose existence is guaranteed by Theorem 5. If the endpoints of $C$ belong to $\partial \mathcal{R}$, then $C$ separates $\mathcal{R}$ into two connected components, namely

$$W_- := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \preceq_{se} y\} \quad \text{and} \quad W_+ := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \preceq_{se} x\},$$

such that the following statements are true.
(i) $\mathcal{W}_-$ is invariant, and $\text{dist}(T^n(x), \mathcal{Q}_2(x)) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\text{dist}(T^n(x), \mathcal{Q}_4(x)) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

If, in addition, $x$ is an interior point of $\mathcal{R}$ and $T$ is $C^2$ and strongly competitive in a neighborhood of $x$, then $T$ has no periodic points in the boundary of $\mathcal{Q}_1(x) \cup \mathcal{Q}_3(x)$ except for $x$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} \mathcal{Q}_2(x)$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int} \mathcal{Q}_4(x)$ for $n \geq n_0$.

In this paper we study the global dynamics of the two rational systems of difference equations mentioned earlier, where all parameters are positive numbers and the initial conditions $x_0$ and $y_0$ are arbitrary nonnegative numbers and $x_0 + y_0 > 0$. In general these systems have the common feature that the global stable manifold of saddle points or non-hyperbolic equilibrium points serve as boundaries of basins of attraction of different local attractors or points at infinity. Another common feature is the existence of a continuum of non-hyperbolic equilibrium points, in which case there exists a critical equilibrium point such that to the right of this point every equilibrium has its stable manifold and to the left, no non-hyperbolic equilibrium point has a basin of attraction.

2 System (14,15)

Now we consider the system (1) where the parameters $A_1, B_2, \beta_1, \gamma_2$ are positive numbers and the initial conditions satisfy $x_0 \geq 0, y_0 \geq 0, x_0 + y_0 > 0$.

The Jacobian matrix of the corresponding map $T(x, y) = \left( \frac{\beta_1 x}{A_1 + y}, \frac{\gamma_2 y}{B_2 x + y} \right)$ is

$$J_T(x, y) = \begin{bmatrix} \frac{\beta_1}{A_1 + y} & \frac{\gamma_2 y}{B_2 x + y} \\ -\frac{\beta_1 x}{(A_1 + y)^2} & \frac{\gamma_2}{(B_2 x + y)^2} \end{bmatrix}$$

and so $\det J_T(x, y) = \frac{A_1 B_2 \beta_1 \gamma_2 x}{(A_1 + y)^2 (B_2 x + y)^2}$.

2.1 Basic Properties of (1)

The next result gives some basic properties of the solutions of the system (1) which play an important role in describing the global dynamics of this system.

**Lemma 1**

(i) System (1) satisfies $(O+)$. Therefore all solutions are eventually componentwise monotone.

(ii) $T$ is injective.

(iii) Every solution $\{(x_n, y_n)\}$ of the system (1) satisfies: $y_n \leq \gamma_2, n \geq 1$.

(iv) Every solution $\{(x_n, y_n)\}$ of the system (1) satisfies: $x_{n+1} \geq \frac{\beta_1}{A_1 + \gamma_2} x_n, n \geq 1$

(v) If $x_0 = 0$ then $(x_n, y_n) = (0, \gamma_2)$ for $n \geq 1$.
(vi) If \( y_0 = 0 \) then \( (x_n, y_n) = \left( \left( \frac{\beta_1}{A_1} \right)^n x_0, 0 \right) \) for \( n \geq 1 \).

(vii) Assume that

\[ \beta_1 \geq A_1 + \gamma_2. \]

Then every solution \( \{(x_n, y_n)\} \) of the system (1) with \( x_0 > 0 \) satisfies

\[ \lim_{n \to \infty} x_n = \infty, \quad \lim_{n \to \infty} y_n = 0. \]

Proof. All properties except (i), (ii) and (vii) follow directly from the equations of the system (1). First, we will prove (i). Now

\[ T \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right) \preceq ne \iff \begin{cases} \frac{\beta_1 x_1}{A_1 + y_1} \leq \frac{\beta_1 x_2}{A_1 + y_2} \\ \frac{\gamma_2 y_1}{B_2 x_1 + y_1} \leq \frac{\gamma_2 y_2}{B_2 x_2 + y_2} \end{cases} \]

\[ \iff \begin{cases} A_1 (x_1 - x_2) \leq x_2 y_1 - x_1 y_2 \\ B_2 (x_2 y_1 - x_1 y_2) \leq 0 \end{cases} \implies x_2 y_1 - x_1 y_2 \leq 0 \implies x_1 \leq x_2 \implies y_1 \leq y_2 \]

\[ \implies \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right) \preceq ne \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right). \]

Therefore, in view of Theorem 2 every solution of the system (1) is eventually component-wise monotonic.

Now we will prove that the \( T \) is injective. Indeed,

\[ T \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right) = T \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \iff \begin{cases} \frac{\beta_1 x_1}{A_1 + y_1} = \frac{\beta_1 x_2}{A_1 + y_2} \\ \frac{\gamma_2 y_1}{B_2 x_1 + y_1} = \frac{\gamma_2 y_2}{B_2 x_2 + y_2} \end{cases} \]

which is equivalent to

\[ \begin{cases} A_1 (x_1 - x_2) = x_2 y_1 - x_1 y_2 \\ B_2 (x_2 y_1 - x_1 y_2) = 0 \end{cases}. \]

This implies

\[ x_2 y_1 - x_1 y_2 = 0 \implies x_1 = x_2 \implies y_1 = y_2. \]

Finally we prove (vii). If \( \beta_1 \geq A_1 + \gamma_2 \) then \( \{x_n\} \) with \( x_0 > 0 \) is a nondecreasing sequence. If \( \lim_{n \to \infty} x_n = L < \infty \), then \( \{(x_n, y_n)\} \) converges to an equilibrium which is impossible. \( \blacksquare \)

2.2 Linearized Stability Analysis

Equilibrium points \((\overline{x}, \overline{y})\) of the system (1) satisfy

\[ \overline{x} = \frac{\beta_1 \overline{x}}{A_1 + \overline{y}}, \quad \overline{y} = \frac{\gamma_2 \overline{y}}{B_2 \overline{x} + \overline{y}} \]

from which we obtain:
Proposition 8 i) If $0 < \beta_1 - A_1 < \gamma_2$, then the system (1) has two equilibrium points: 
$E_1 = (0, \gamma_2)$ and $E_2 = \left( \frac{\gamma_2 + A_1 - \beta_1}{B_2}, \beta_1 - A_1 \right)$.

ii) If $\gamma_2 \leq \beta_1 - A_1$ or $\beta_1 < A_1$, then the system (1) has the unique equilibrium point: 
$E_1 = (0, \gamma_2)$.

iii) If $\beta_1 = A_1$, then the system (1) has the following equilibrium points: 
$E_1 = (0, \gamma_2)$ and $E_x = (\bar{x}, 0)$, for all $\bar{x} > 0$.

The next result follows from the linearized stability analysis of the system (1).

Lemma 2 i) If $\gamma_2 > \beta_1 - A_1$, then the equilibrium point $E_1 = (0, \gamma_2)$ of the system (1) is locally asymptotically stable.

ii) If $\gamma_2 < \beta_1 - A_1$, then the equilibrium point $E_1 = (0, \gamma_2)$ of the system (1) is a saddle point.

iii) If $\gamma_2 = \beta_1 - A_1 > 0$, then the equilibrium point $E_1 = (0, \gamma_2)$ of the system (1) is non-hyperbolic with $\lambda_1 = 0 < 1$ i $\lambda_2 = 1$.

iv) If $0 < \beta_1 - A_1 < \gamma_2$, then the equilibrium point $E_2 = \left( \frac{\gamma_2 + A_1 - \beta_1}{B_2}, \beta_1 - A_1 \right)$ of the system (1) is a saddle point.

v) If $\beta_1 = A_1$, then the equilibrium points $E_x = (\bar{x}, 0)$, $\bar{x} > 0$ of the system (1) are non-hyperbolic.

Proof. i)-iii) By (9) and (6) we have that Jacobian matrix evaluated at the equilibrium point $E_1$ has the form

$$J_T(E_1) = \begin{bmatrix} \frac{\beta_1}{A_1 + \gamma_2} & 0 \\ -B_2 & 0 \end{bmatrix},$$

and the corresponding characteristic equation has roots

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\beta_1}{A_1 + \gamma_2}. \quad (10)$$

From (10) we see that all claims i)-iii) in the Lemma hold.

iv) The Jacobian matrix of the map $T(x, y) = \left( \frac{\beta_1 x}{A_1 + y}, \frac{\gamma_2 y}{B_2 x + y} \right)$ of the equilibrium point $E_2$ is of the form

$$J_T(E_2) = \begin{bmatrix} \frac{1}{-B_2(\beta_1 - A_1)} & -\frac{\gamma_2 + A_1 - \beta_1}{\gamma_2 + A_1 - \beta_1} \\ \frac{-B_2(\beta_1 - A_1)}{\gamma_2 + A_1 - \beta_1} & 0 \end{bmatrix}.$$

The corresponding characteristic equation evaluated at the equilibrium point $E_2$ is

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2}, \quad q = \frac{A_1(\gamma_2 + A_1 - \beta_1)}{\gamma_2 \beta_1}.$$

By $0 < \beta_1 - A_1 < \gamma_2$ we have $p > 0$ and $q > 0$ so we need to show

(I) $p > 1 + q$ and (II) $p^2 - 4q > 0$. 
Indeed

\( (I) \quad p > 1 + q \iff 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2} > 1 + \frac{A_1 (\gamma_2 + A_1 - \beta_1)}{\gamma_2 \beta_1} \iff \beta_1 - A_1 > 0, \)

which is satisfied.

Next

\( (II) \quad p^2 - 4q > 0 \iff \left( 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2} \right)^2 - 4 \frac{A_1 (\gamma_2 + A_1 - \beta_1)}{\gamma_2 \beta_1} \)

\[ \iff \frac{\beta_1 - A_1}{\beta_1} \left( \beta_1 (\beta_1 - A_1) + 4 \gamma_2 (\gamma_2 + A_1 - \beta_1) \right) > 0, \]

which is true because \( \gamma_2 > \beta_1 - A_1 > 0. \)

\( v) \) The Jacobian matrix of the map \( T(x, y) = \left( \frac{\beta_1 x}{x + y}, \frac{\gamma_2 y}{B_2 x + y} \right) \) of the system (1) at the equilibrium point \( E_\overline{x} \) is of the form

\[
J_T(E_2) = \begin{bmatrix}
1 & -\frac{\overline{x}}{\beta_1} \\
0 & \frac{\gamma_2}{B_2 \overline{x}}
\end{bmatrix}.
\]

The eigenvalues are

\( \lambda_1 = 1 \) and \( \lambda_2 = \frac{\gamma_2}{B_2 \overline{x}}. \)

Notice that

1. \( 0 < \lambda_2 < 1 \iff \overline{x} > \frac{\gamma_2}{B_2}, \)
2. \( \lambda_2 > 1 \iff \overline{x} < \frac{\gamma_2}{B_2}. \)
3. \( \lambda_2 = 1 \iff \overline{x} = \frac{\gamma_2}{B_2}. \)

In all cases the corresponding eigenvector is \( v = (\overline{x}, (1 - \lambda_2)A_1). \)

\section{2.3 Global results}

In this section we characterize the global behavior of the system (1) for all values of parameters.

\textbf{Theorem 9} Consider the system (1) and assume that \( 0 < \beta_1 - A_1 < \gamma_2. \) Then there exists a set \( C \subset \mathcal{R} \) which is invariant and a subset of the basin of attraction of \( E_2. \) The set \( C \) is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates \( \mathcal{R} \) into two connected and invariant components, namely

\[
W_- := \{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \leq_{se} y \} \quad \text{and} \quad W_+ := \{ x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \leq_{se} x \}.
\]

Solutions \( \{ (x_n, y_n) \} \) to (1) with \( (x_0, y_0) \notin C \) satisfy

\[
\lim_{n \to \infty} (x_n, y_n) = (\infty, 0) \text{ for every } (x_0, y_0) \in W_+,
\]

and

\[
\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2) \text{ for every } (x_0, y_0) \in W_-.
\]
Proof. Clearly, the system (1) is strongly competitive on $\mathcal{R} = (0, \infty) \times (0, \infty)$. Thus, all conditions of Theorems 5, 6 and 7 and Corollary 1 are satisfied and the conclusion of the theorem follows. ■

**Theorem 10** Assume that $\gamma_2 \leq \beta_1 - A_1$.

i. If $x_0 \neq 0$, then $\lim_{n \to \infty} x_n = \infty$, $\lim_{n \to \infty} y_n = 0$.

ii. If $x_0 = 0$, then $x_n = 0$, $\lim_{n \to \infty} y_n = \gamma_2$, i.e. the $y$-axis is the stable manifold of the equilibrium point $E_1 = (0, \gamma_2)$.

Proof. It follows from Lemma 1. ■

**Theorem 11** Consider the system (1) and assume that $\beta_1 < A_1$. Then the equilibrium point $E_1 = (0, \gamma_2)$ is globally asymptotically stable, i.e.

$$\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2)$$

for every $(x_0, y_0), x_0 \geq 0, y_0 \geq 0$ such that $x_0 + y_0 > 0$.

Proof. We have

$$x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n} < \frac{\beta_1}{A_1} x_n \Rightarrow x_n < \left(\frac{\beta_1}{A_1}\right)^n x_0 \Rightarrow \lim_{n \to \infty} x_n = 0.$$  

Because the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are eventually monotone and $y_n \leq \gamma_2$, for all $n \geq 1$, we see (from the second equation of the system (1)) that $\lim_{n \to \infty} y_n = \gamma_2$. ■

**Theorem 12** Consider the system (1) and assume that $\beta_1 = A_1$. Then $\{x_n\}$ is decreasing and so is convergent. Every solution $\{(x_n, y_n)\}$ of the system (1) converges to an equilibrium point. More precisely,

i. If $\bar{x} > \frac{\gamma_2}{B_2}$, then there exists a set $\mathcal{W}^s(E_x) \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of $E_x$. The set $\mathcal{W}^s(E_x)$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold).

(ii) If $\bar{x} \leq \frac{\gamma_2}{B_2}$, then all solutions which start to the left of $\frac{\gamma_2}{B_2}$ are attracted to $E_1$, that is $\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2)$ for every $(x_0, y_0)$ such that $0 < x_0 \leq \frac{\gamma_2}{B_2}$.

Proof. Notice that the condition $\beta_1 = A_1$ implies that the sequence $\{x_n\}$ is decreasing and so is convergent. Thus in this case all solutions of the system (1) are bounded, in such a way that the bound on $\{y_n\}_{n=1}^{\infty}$ is uniform and that the bound on $\{x_n\}_{n=1}^{\infty}$ is solution dependent.

Assume that $\bar{x} > \frac{\gamma_2}{B_2}$. Then, in view of Lemma 2 the second eigenvalue is in $(0, 1)$ and the corresponding eigenvector $v = (\bar{x}, (1 - \lambda_2)A_1)$ is pointed towards the first quadrant. Thus all conditions of Theorem 5 are satisfied and so the existence of invariant curves emanating from $E_x$ is guaranteed, which proves (i). Let us choose $(x, y)$ such that $y > 0$. Now we have

$$T((x, y)) - (x, y) = \left(-\frac{xy}{A_1 + y}, y, \frac{\gamma_2 - B_2 x - y}{B_2 x + y}\right)$$
which implies that $T((x, y)) \preceq_{ne} (x, y)$ if $x > \frac{\gamma_2}{B_2}$ and $T((x, y)) \preceq_{se} (x, y)$ if $x < \frac{\gamma_2}{B_2}$ and $y$ is small enough. Furthermore, it is clear that the region $L = \{(x, y): x \leq \frac{\gamma_2}{B_2}\}$ is invariant. Thus, if $(x_0, y_0) \in L$ then either $T((x_0, y_0)) \preceq_{se} (x_0, y_0)$ or $T((x_0, y_0)) \preceq_{ne} (x_0, y_0)$. In the first case we would get the monotonically decreasing sequence $\{T^n((x_0, y_0))\}$ which is bounded below by $(0, \gamma_2)$. Thus, this sequence must be convergent and the only equilibrium to which it can converge is $E_1$. In the second case, either $\{T^n((x_0, y_0))\}$ is an infinite sequence of ordered points in the North-East ordering or there exists some $N$ such that $T((x_N, y_N)) \preceq_{se} (x_N, y_N)$. The first case is not possible because such a sequence would converge to some equilibrium point $(\bar{x}, 0), \bar{x} < \frac{\gamma_2}{B_2}$, which is, in view of $T((x, y)) \preceq_{se} (x, y)$ if $x < \frac{\gamma_2}{B_2}$ and $y$ is small enough, impossible. Thus $T((x_N, y_N)) \preceq_{se} (x_N, y_N)$ for some $N$, which leads to the existence of a monotonically decreasing sequence $\{T^n((x, y))\}_{n \geq N}$ which converges to $E_1$.

If $(x_0, y_0), x_0 > \frac{\gamma_2}{B_2}$ then $T((x_0, y_0)) \preceq_{ne} (x_0, y_0)$, in which case we get that either $\{T^n((x_0, y_0))\}$ is an infinite sequence of ordered points in the North-East ordering or there will exist some $N$ such that $T((x_N, y_N)) \preceq_{se} (x_N, y_N)$. If such an $N$ exists then the corresponding solution converges to $E_1$. Otherwise, the solution converges to one of the equilibrium points $E_x$ for $x \geq \frac{\gamma_2}{B_2}$.

If $(x_0, y_0) = \left(\frac{\gamma_2}{B_2}, y_0\right), y_0 > 0$ then $0 < x_1 < x_0, 0 < y_1 < y_0$. Thus, the corresponding solution converges to $E_1$. $\blacksquare$

The obtained results lead to the following characterization of the boundedness of solutions of the system (1).

**Corollary 2** Every solution $\{(x_n, y_n)\}$ of the system (1) is bounded if and only if $\beta_1 \leq A_1$. More precisely, $\{x_n\}$ is bounded if and only if $\beta_1 \leq A_1$ and $\{y_n\}$ is always bounded.

### 3 System (14,38)

Now we consider the system (2) where the parameters $A_1, A_2, \beta_1, B_2$ and $\gamma_2$ are positive numbers and the initial conditions satisfy $x_0 \geq 0, y_0 \geq 0, x_0 + y_0 \geq 0$.

The Jacobian matrix of the corresponding map $T(x, y) = \left(\frac{\beta_1 x}{A_1+y}, \frac{\gamma_2 y}{A_2+B_2 x+y}\right)$ is of the form

$$J_T(x, y) = \begin{bmatrix} \frac{\beta_1}{A_1+y} & -\frac{\beta_1 x}{(A_2+B_2 x+y)^2} \\ -\frac{\gamma_2 y}{B_2 x+y} & \frac{\gamma_2 (A_2+B_2 x+y)}{(A_2+B_2 x+y)^2} \end{bmatrix},$$

and so $\det J_T(x, y) = \beta_1 \gamma_2 A_1 A_2 + A_1 B_2 x + A_2 y (A_1+y)^2 (A_2+B_2 x+y)^2 > 0$.

#### 3.1 Basic Properties of (2)

The next result gives some basic properties of the solutions of the system (2) which play an important role in describing the global dynamics of this system.

**Lemma 3** (i) $T$ is injective.

(ii) System (2) satisfies $(O+).$ Therefore all solutions are eventually componentwise monotone.
(iii) Every solution \( \{(x_n, y_n)\} \) of the system (2) satisfies: \( y_n < \gamma_2, \ n \geq 1 \).

(iv) Every solution \( \{(x_n, y_n)\} \) of the system (1) satisfies: \( x_{n+1} \geq \frac{\beta_1}{A_1 + \gamma_2} x_n, \ n \geq 1 \).

(v) If \( x_0 = 0 \) then \( x_n = 0 \) and \( y_n = \left( \frac{1}{\gamma_2 - A_2} + \frac{1}{\gamma_0} \right) \left( \frac{A_2}{\gamma_2} \right)^n \) if \( \gamma_2 \neq A_2 \), and
\[
y_n = \left( \frac{n}{A_2} + \frac{1}{\gamma_0} \right)^{-1} \text{ if } \gamma_2 = A_2, \text{ for } n \geq 1.
n
(vi) If \( y_0 = 0 \) then \( (x_n, y_n) = \left( \left( \frac{\beta_1}{A_1} \right)^n x_0, 0 \right) \) for \( n \geq 1 \).

(vii) Assume that
\[
\beta_1 \geq A_1 + \gamma_2.
\]
Then every solution \( \{(x_n, y_n)\} \) of the system (2) with \( x_0 > 0 \) satisfies (8).

**Proof.** All properties except (i), (ii), (v) and (vii) follow directly from the equations of the system (2). First, we will prove (i). Now
\[
T \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]
implies
\[
A_1 (x_1 - x_2) = x_2 y_1 - x_1 y_2, \quad A_2 (y_1 - y_2) = B_2 (x_1 y_2 - x_2 y_1).
\]
If \( x_1 = x_2 \) then clearly \( y_1 = y_2 \). Assume \( x_1 \neq x_2 \) and specifically \( x_1 > x_2 \). Then \( x_2 y_1 - x_1 y_2 > 0 \), which implies \( y_1 > y_2 \) and by the second equality above \( x_1 y_2 - x_2 y_1 > 0 \), which is a contradiction.

Second, we will prove (ii). Actually, (ii) is proved by applying Theorem 3 and observing that \( T \) is in view of (i) injective, and the fact that \( \det J_T(x, y) > 0 \).

Therefore, in view of Theorem 2 every solution of the system (2) is eventually component-wise monotonic.

Next we prove (v). If \( x_0 = 0 \) then \( x_n = 0 \) and the second equation of the system (2) becomes
\[
y_{n+1} = \frac{\gamma_2 y_n}{A_2 + y_n},
\]
which is Riccati’s equation with a known solution, see [9, 10].

Finally, we prove (vii). If \( \beta_1 \geq A_1 + \gamma_2 \) then \( \{x_n\} \) with \( x_0 > 0 \) is a nondecreasing sequence. If \( \lim_{n \to \infty} x_n = L < \infty \), then \( \{x_n, y_n\} \) converges to a finite equilibrium which is impossible.

3.2 Linearized Stability Analysis

Equilibrium points \( (\mathcal{X}, \mathcal{Y}) \) of the system (2) are solutions of the system of equations
\[
\mathcal{X} = \frac{\beta_1 \mathcal{X}}{A_1 + \mathcal{Y}}, \quad \mathcal{Y} = \frac{\gamma_2 \mathcal{Y}}{A_2 + B_2 \mathcal{X} + \mathcal{Y}}
\]
from which we obtain:
Proposition 13  System (2) has the following equilibrium points:

(i) \( E_0(0,0) \)

(ii) If \( A_1 < \beta_1 < A_1 + \gamma_2 - A_2 \), then the system (2) has the equilibrium point:
\[
E_2 = \left( \frac{\gamma_2 - A_1 - \beta_1 - A_2}{B_2}, \beta_1 - A_1 \right).
\]

(iii) If \( \gamma_2 > A_2 \), then the system (2) has the equilibrium point: \( E_y = (0, \gamma_2 - A_2) \).

(iv) If \( \beta_1 = A_1 \), then the system (2) has the following equilibrium points:
\[
E_x = (x, 0), \text{ for all } x > 0.
\]

The next result follows from the linearized stability analysis of the system (2).

Lemma 4  The equilibrium point \( E_0 \) of the system (2) is:

(i) locally asymptotically if \( \beta_1 < A_1, \gamma_2 < A_2 \).

(ii) a repeller if \( \beta_1 > A_1, \gamma_2 > A_2 \).

(iii) a saddle point if \( \beta_1 < A_1, \gamma_2 > A_2 \) or \( \beta_1 > A_1, \gamma_2 < A_2 \).

(iv) non-hyperbolic if \( \beta_1 = A_1 \) and/or \( \gamma_2 = A_2 \).

Proof. The proof follows from the fact that the Jacobian matrix evaluated at the equilibrium point \( E_0 \) has the form
\[
J_T(E_0) = \begin{bmatrix}
\frac{\beta_1}{A_1} & 0 \\
\frac{\gamma_2}{A_2} & \frac{\gamma_2}{A_2}
\end{bmatrix}.
\]

Lemma 5  Assume that \( \gamma_2 > A_2 \). The equilibrium \( E_y \) of the system (2) is:

(i) locally asymptotically if \( \beta_1 < A_1 + \gamma_2 - A_2 \).

(ii) a saddle point if \( \beta_1 > A_1 + \gamma_2 - A_2 \).

(iii) non-hyperbolic if \( \beta_1 = A_1 + \gamma_2 - A_2 \).

Proof. The proof follows from the fact that the Jacobian matrix evaluated at the equilibrium point \( E_y \) has the form
\[
J_T(E_y) = \begin{bmatrix}
\frac{\beta_1}{A_1 + \gamma_2 - A_2} & 0 \\
\frac{\gamma_2}{A_2 (\gamma_2 - A_2)} & \frac{A_2}{\gamma_2}
\end{bmatrix}.
\]

Lemma 6  Assume that \( A_1 < \beta_1 < A_1 + \gamma_2 - A_2 \). The equilibrium \( E_2 \) of the system (2) is a saddle point.
Proof. The Jacobian matrix of the map \( T(x, y) = \left( \frac{\beta_1 x}{A_1 + y}, \frac{\gamma_2 y}{B_2 x + y} \right) \) of the equilibrium point \( E_2 \) is of the form
\[
J_T(E_2) = \begin{bmatrix}
\frac{1}{\gamma_2} & -\frac{\beta_2 + A_1 - \beta_1 - A_2}{\gamma_2}
\frac{-\beta_2 A_1}{\gamma_2^2} & \frac{\beta_2 B_2}{\gamma_2^2}
\end{bmatrix}.
\]
The corresponding characteristic equation evaluated at the equilibrium point \( E_2 \) is
\[
\lambda^2 - p\lambda + q = 0,
\]
where
\[
p = 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2},
q = \frac{\beta_1 A_2 + A_1 (\gamma_2 + A_1 - \beta_1 - A_2)}{\gamma_2 \beta_1}.
\]
By \( 0 < \beta_1 - A_1 < \gamma_2 \) and \( \beta_1 < A_1 + \gamma_2 - A_2 \) we have \( p > 0 \) and \( q > 0 \) so we need to show

(I) \( p > 1 + q \) and (II) \( p^2 - 4q > 0 \).

Indeed

(I) \( p > 1 + q \Longleftrightarrow 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2} > 1 + \frac{\beta_1 A_2 + A_1 (\gamma_2 + A_1 - \beta_1 - A_2)}{\gamma_2 \beta_1} \)

\( \Longleftrightarrow (\beta_1 - A_1) (\gamma_2 + A_1 - \beta_1 - A_2) > 0 \)

which is satisfied.

Next

(II) \( p^2 - 4q > 0 \Longleftrightarrow \left( 1 + \frac{\gamma_2 + A_1 - \beta_1}{\gamma_2} \right)^2 - 4 \frac{\beta_1 A_2 + A_1 (\gamma_2 + A_1 - \beta_1 - A_2)}{\gamma_2 \beta_1} \)

\( \Longleftrightarrow (\beta_1 - A_1) (\beta_1 (\beta_1 - A_1) + 4\gamma_2 (\gamma_2 + A_1 - \beta_1 - A_2)) > 0, \)

which, in view of our conditions, holds. \( \blacksquare \)

Lemma 7 Assume that \( \beta_1 = A_1 \). The equilibrium \( E_x \) of the system (2) is a non-hyperbolic equilibrium point with \( \lambda_1 = 1 \) and \( \lambda_2 = \frac{\gamma_2}{A_2 + B_2 \bar{x}} \).

Proof. The proof follows from the fact that the Jacobian matrix evaluated at the equilibrium point \( E_x \) has the form
\[
J_T(E_x) = \begin{bmatrix}
1 & -\frac{1}{A_2 + B_2 \bar{x}}
0 & \frac{1}{A_2 + B_2 \bar{x}}
\end{bmatrix}.
\]

Note that

(1) \( \lambda_2 < 1 \Longleftrightarrow \bar{x} > \frac{\gamma_2 - A_2}{B_2}, \)
(2) \( \lambda_2 > 1 \Longleftrightarrow \bar{x} < \frac{\gamma_2 - A_2}{B_2} \)
(3) \( \lambda_2 = 1 \Longleftrightarrow \bar{x} = \frac{\gamma_2 - A_2}{B_2}, \)

In all cases the corresponding eigenvector is \( \mathbf{v} = (\bar{x}, (1 - \lambda_2) A_1). \) \( \blacksquare \)
3.3 Global results

In this section we characterize the global behavior of the system (2) for all values of parameters.

**Theorem 14** Consider the system (2) and assume that $\beta_1 < A_1$, $\gamma_2 \leq A_2$. Then the unique equilibrium point $E_0$ is globally asymptotically stable.

**Proof.** By Lemma 4 $E_0$ is locally asymptotically stable. Further we have

$$x_{n+1} < \frac{\beta_1}{A_1} x_n, \quad y_{n+1} < \frac{\gamma_2}{A_2} y_n, \quad n \geq 1,$$

which implies

$$x_n < \left(\frac{\beta_1}{A_1}\right)^n x_0, \quad y_{n+1} < y_n, \quad n \geq 1,$$

which shows that

$$\lim_{n \to \infty} x_n = 0, \quad \lim_{n \to \infty} y_n = 0.$$

If $\gamma_2 = A_2$ then $\{y_n\}$ is strictly decreasing and so is convergent. Since there is only one equilibrium point $E_0$ we have $\lim_{n \to \infty} y_n = 0$. ■

**Theorem 15** Consider the system (2) and assume that $A_1 < \beta_1 < A_1 + \gamma_2 - A_2$. Then the system (2) has three equilibrium points $E_0$, $E_y$, and $E_2$. Then there exists a set $C \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of $E_2$. The set $C$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) with an end point at $E_0$ and separates $\mathcal{R}$ into two connected and invariant components, namely

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \preceq y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \preceq x\}.$$

Solutions $\{(x_n, y_n)\}$ to (2) with $(x_0, y_0) \notin C$ satisfy

$$\lim_{n \to \infty} (x_n, y_n) = (\infty, 0) \text{ for every } (x_0, y_0) \in \mathcal{W}_+,$$

and

$$\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2 - A_2) \text{ for every } (x_0, y_0) \in \mathcal{W}_-.$$

**Proof.** Clearly, the system (2) is strongly competitive on $\mathcal{R} = (0, \infty) \times (0, \infty)$. In view of Lemmas 3 and 6, we see that all conditions of Theorems 5, 6 and 7 and Corollary 1 are satisfied and the conclusion of the theorem follows. ■

**Theorem 16** Assume that $\beta_1 \geq A_1 + \gamma_2$. Then the system (2) has either one equilibrium point $E_0$, when $\gamma_2 \leq A_2$ or two equilibrium points $E_0$ and $E_y$ when $\gamma_2 > A_2$.

i. If $x_0 \neq 0$, then $\lim_{n \to \infty} x_n = \infty, \quad \lim_{n \to \infty} y_n = 0$.

ii. If $x_0 = 0$, then $\lim_{n \to \infty} (x_n, y_n) = E_y$, when $\gamma_2 > A_2$ and $\lim_{n \to \infty} (x_n, y_n) = E_0$, when $\gamma_2 \leq A_2$.

**Proof.** It follows from Lemma 3. ■
Theorem 17 Consider the system (2) and assume that \( \beta_1 < A_1, \gamma_2 > A_2 \). Then the system (2) has two equilibrium points \( E_0 \) and \( E_y \) and the equilibrium point \( E_y = (0, \gamma_2 - A_2) \) is globally asymptotically stable, i.e.

\[
\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2 - A_2)
\]

for every \( (x_0, y_0) \), \( x_0 \geq 0 \), such that \( y_0 \geq 0, x_0 + y_0 > 0 \).

Proof. The first equation of the system (2) implies

\[
x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n} < \frac{\beta_1}{A_1} x_n =: x_n \Rightarrow x_n \leq \left( \frac{\beta_1}{A_1} \right)^n x_0 \Rightarrow \lim_{n \to \infty} x_n = 0.
\]

Because the sequences \( \{x_n\}_{n=0}^{\infty} \) and \( \{y_n\}_{n=0}^{\infty} \) are eventually monotone and \( y_n \leq \gamma_2 \), for all \( n \geq 1 \), we see from the second equation of the system (2) that \( \lim_{n \to \infty} y_n = \gamma_2 - A_2 \). \( \blacksquare \)

Theorem 18 Consider the system (2) and assume that \( \beta_1 = A_1, \gamma_2 > A_2 \). Then \( \{x_n\} \) is decreasing and so is convergent. Every solution \( \{(x_n, y_n)\} \) of the system (2) converges to an equilibrium point. More precisely,

(i) If \( \bar{x} > \frac{\gamma_2 - A_2}{B_2} \), then there exists a set \( \mathcal{W}^s(E_x) \subset \mathcal{R} \) which is invariant and a subset of the basin of attraction of \( E_x \). The set \( \mathcal{W}^s(E_x) \) is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold).

(ii) If \( \bar{x} \leq \frac{\gamma_2 - A_2}{B_2} \), then all solutions which start to the left of \( \frac{\gamma_2 - A_2}{B_2} \) are attracted to \( E_y \), that is

\[
\lim_{n \to \infty} (x_n, y_n) = (0, \gamma_2 - A_2) \text{ for every } (x_0, y_0) \text{ such that } 0 < x_0 < \frac{\gamma_2 - A_2}{B_2}.
\]

Proof. Observe that the condition \( \beta_1 = A_1 \) implies that the sequence \( \{x_n\} \) is decreasing and so is convergent. Thus in this case all solutions of the system (2) are bounded, in such a way that the bound on \( \{y_n\}_{n=1}^{\infty} \) is uniform and that the bound on \( \{x_n\}_{n=1}^{\infty} \) is solution dependent. Assume that \( \bar{x} > \frac{\gamma_2 - A_2}{B_2} \). Then, in view of Lemma 3 the second eigenvalue is in \( (0, 1) \) and the corresponding eigenvector \( \mathbf{v} = (\bar{x}, (1 - \lambda_2)A_1) \) is pointed toward the first quadrant. Thus all conditions of Theorem 5 are satisfied and so the existence of invariant curves emanating from \( E_x \) is guaranteed, which proves (i).

Let us choose \( (x, y) \) such that \( y > 0 \). Now we have

\[
T((x,y)) - (x,y) = \left( -\frac{xy}{A_1 + y}, y\frac{\gamma_2 - A_2 - B_2x - y}{A_2 + B_2x + y} \right)
\]

which implies that \( T((x,y)) \leq_{nc} (x,y) \) if \( x > \frac{\gamma_2 - A_2}{B_2} \) and \( T((x,y)) \leq_{se} (x,y) \) if \( x < \frac{\gamma_2 - A_2}{B_2} \) and \( y \) is small enough. The rest of this proof is similar to the proof of Theorem 12, where \( \frac{\gamma_2 - A_2}{B_2} \) should be replaced with \( \frac{\gamma_2 - A_2}{B_2} \) and will be omitted.

If \( (x_0, y_0) = \left( \frac{\gamma_2 - A_2}{B_2}, y_0 \right), y_0 > 0 \) then \( 0 < x_1 < x_0, 0 < y_1 < y_0 \). Thus, the corresponding solution converges to \( E_y \). \( \blacksquare \)
Theorem 19 Consider the system (2) and assume that $\beta_1 = A_1, \gamma_2 \leq A_2$. Then every point of the $x$-axis is a non-hyperbolic equilibrium point and every solution $\{(x_n, y_n)\}$ of the system (2) converges to an equilibrium point. More precisely, there exists a set $W^s(E_x) \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of $E_x$. The set $W^s(E_x)$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold).

Proof. The proof is basically the same as the proof of (i) of Theorem 18 and so it will be omitted. ■

Theorem 20 Consider the system (2) and assume that $\beta_1 > A_1, \gamma_2 \leq A_2$. Then the system (2) has one equilibrium point $E_0$ which is a saddle point if $\gamma_2 < A_2$ and a non-hyperbolic equilibrium point if $\gamma_2 = A_2$. In either case all solutions of the system (2) satisfy

$$\lim_{n \to \infty} (x_n, y_n) = (\infty, 0)$$

for every $(x_0, y_0)$ such that $x_0 > 0$.

Proof. The second equation of (2) implies that $\{y_n\}$ is strictly decreasing and so it must converge to 0. Choose $\epsilon > 0$ and $N$ such that $y_n < \epsilon$ for $n \geq N$ and that $\beta_1 > A_1 + \epsilon$. Then the first equation of (2) implies that

$$x_{n+1} > \frac{\beta_1}{A_1 + \epsilon} x_n, \quad n \geq N$$

and so $\lim_{n \to \infty} x_n = \infty$. ■

Theorem 21 Consider the system (2) and assume that $\gamma_2 > A_2, A_1 + \gamma_2 - A_2 < \beta_1 < A_1 + \gamma_2$. Then there exist two equilibrium points $E_0, E_y$. Every solution with $x_0, y_0 > 0$ satisfies

$$\lim_{n \to \infty} (x_n, y_n) = (\infty, 0).$$

Proof. Since $\{y_n\}$ is a bounded and monotone sequence it must be convergent. Thus $\lim_{n \to \infty} y_n = 0$. Choose $N$ large enough such that $y_n < \gamma_2 - A_2$ for $n \geq N$. Then the first equation of (2) implies that

$$x_{n+1} > \frac{\beta_1}{A_1 + \gamma_2 - A_2} x_n, \quad n \geq N$$

and so $\lim_{n \to \infty} x_n = \infty$. ■

References


