MTH 244 - Additional Information for Chapter 3

Section 1 (Merino) and section 3 (Dobrushkin) - March 2003

1 Linear Systems of Differential Equations of Order One

A system of n first order linear differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n + b_1(t) \\ &\vdots \\ x_1' &= a_{n1}x_1 + \cdots + a_{nn}x_n + b_n(t) \end{aligned}$$

may be written in matrix form as

$$\mathbf{x}' = A(t)\mathbf{x} + b(t)$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

The system is said to be *homogeneous* if $b(t) =$ the zero vector, and the system has constant coefficients if the a_{ij}'s and the b_j's do not depend on the independent variable t.

Therefore a homogeneous linear system with constant coefficients has the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is a constant, $n \times n$ matrix.

Example 1. Consider the system

$$\begin{aligned} x_1' &= 5x_1 + 3x_2 \\ x_2' &= -6x_1 - 4x_2 \end{aligned}$$

Then, the matrix A is given by

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$$

Later we will see how to solve the system. For now let's take a look at two solutions to the system:

$$\mathbf{x}^1 = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} \qquad \mathbf{x}^2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

You may check that these are solutions by direct substitution. Linear independence can be checked by showing that the equation $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 = \mathbf{0}$ has as unique solution $c_1 = c_2 = 0$. Another method to check linear independence is to form a square matrix with columns $\mathbf{x}^1$ and $\mathbf{x}^2$, then verify that the determinant of the matrix is not zero.

Theorem 1 (Superposition Principle). *If $\mathbf{x}^1$ and $\mathbf{x}^2$ are two solutions to the homogeneous equation $\mathbf{x}' = A(t)\mathbf{x}$, then for any choice of constants c_1 and c_2 the vector function $\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2$ is also a solution.*

We will present later theorems that give the solution to systems of DEs when the matrix A is constant.

2 Matrix Exponential

Recall from Calculus MTH 142 the Taylor series expansion of the exponential function:

$$e^x = I + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

The exponential of a $n \times n$ (square) matrix B may be defined with the same formula:

$$e^B = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{n!}B^n + \cdots$$

Here I is the $n \times n$ identity matrix, and $B^2 = B \cdot B$, $B^3 = BBB$, etc. That the series "converges" is something discussed in more advanced courses. For now we accept this fact. A more rigorous definition of matrix exponential is given in the Appendix.

When dealing with systems of differential equations, one has often to deal with expressions like e^{At}, where A is a matrix and t is a real number or real variable. With the above formula we get

$$e^{At} = I + At + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots + \frac{1}{n!}t^nA^n + \cdots$$

As an interesting observation, note that term by term differentiation of the formula for e^{At} gives that

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Here is a proof of this statement:

$$\begin{aligned}
\tfrac{d}{dt}e^{At} &= 0 + A + \tfrac{1}{2!}2tA^2 + \tfrac{1}{3!}3t^2A^3 + \cdots + \tfrac{1}{n!}nt^{n-1}A^n + \cdots \\
&= 0 + A + \tfrac{1}{1!}tA^2 + \tfrac{1}{2!}t^2A^3 + \cdots + \tfrac{1}{(n-1)!}t^{n-1}A^n + \cdots \\
&= A\left(I + At + \tfrac{1}{2!}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots + \tfrac{1}{n!}t^nA^n + \cdots\right) \\
&= Ae^{At}
\end{aligned}$$

Calculating e^{At} is best done with computers. There are sophisticated algorithms for doing this efficiently.

3 Solving linear systems of DEs when A is constant

Consider the homogeneous system of DEs $\mathbf{x}' = A\mathbf{x}$, with initial condition $\mathbf{x}(0) = \mathbf{x}^0$, and let us form the vector

$$\tilde{\mathbf{x}} = e^{At}\mathbf{x}^0$$

A simple computation gives that

$$\frac{d}{dt}\tilde{\mathbf{x}} = \frac{d}{dt}(e^{At}\mathbf{x}^0) = Ae^{At}\mathbf{x}^0 = A\tilde{\mathbf{x}}$$

hence we conclude that $\tilde{\mathbf{x}}$ is a solution to the system of DEs. Moreover, we have $\tilde{\mathbf{x}}(0) = \mathbf{x}^0$, so $\tilde{\mathbf{x}}$ also satisfies the initial condition. This gives the following result.

Theorem 2. *The solution to the initial value problem*

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\tilde{\mathbf{x}} = e^{At}\mathbf{x}_0$$

The following result gives the solution when the system is not homogeneous.

Theorem 3. *The solution to the initial value problem*

$$\mathbf{x}' = A\mathbf{x} + b(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = e^{At}\mathbf{x}_0 + \int_0^t e^{(t-s)A}b(s)ds$$

Integration of the vector function $e^{(t-s)A}b(s)$ is to be done entry by entry.

WARNING: Theorems 1 and 2do not apply if the matrix A is not constant!

4 Computing the exponential of a matrix with Maple

We load the **linalg** package first and enter a matrix, followed with the command to get e^{Bt}:
> **with(linalg):** > **B:=array([[1,1],[4,1]]);**

$$B := \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

> **EBt :=exponential(B,t);**

$$EBt := \begin{bmatrix} \frac{1}{2}e^{(-t)} + \frac{1}{2}e^{(3\,t)} & \frac{1}{4}e^{(3\,t)} - \frac{1}{4}e^{(-t)} \\ e^{(3\,t)} - e^{(-t)} & \frac{1}{2}e^{(-t)} + \frac{1}{2}e^{(3\,t)} \end{bmatrix}$$

We now calculate the determinant of e^{Bt} to verify that the column vectors are linearly independent.

> **det(EBt);**

$$e^{(-t)}\,e^{(3\,t)}$$

> **simplify(%);**

$$e^{(2\,t)}$$

The determinant of e^{Bt} is nonzero for all t, so its column vectors are linearly independent functions.
Now we show how to define a column vector of constants in Maple. It is displayed as a row to save space:

> **vec := vector([-2,5]);**

$$vec\ :=\ [-2, 5]$$

The next computation shows how to multiply the matrix e^{Bt} by the column vector:

> **Bv := evalm(EBt&*vec);**

$$Bv := \left[-\frac{9}{4}e^{(-t)} + \frac{1}{4}e^{(3\,t)},\ \frac{1}{2}e^{(3\,t)} + \frac{9}{2}e^{(-t)}\right]$$

Here is the first entry of the resulting vector:

> **Bv[1];**

$$-\frac{9}{4}e^{(-t)} + \frac{1}{4}e^{(3\,t)}$$

5 Appendix: Rigorous definition of matrix exponential

The exponential matrix, $\mathbf{X}(t) = e^{\mathbf{A}t}$, may be defined as follows:

$$e^{\mathbf{A}t} = \sum_{j=0}^{m-1} b_j(t)\,\mathbf{A}^j, \tag{1}$$

where m is the dimension of the matrix $\mathbf{A}$, and the coefficient functions $b_j(t)$, $j = 0, 1, \ldots, m-1$, in exponential representation (1) satisfy the following equations:

$$e^{\lambda_k t} = b_0(t) + b_1(t)\lambda_k + \cdots + b_{m-1}(t)\lambda_k^{m-1},\ \ k = 1, 2, \ldots, s. \tag{2}$$

We denote here λ_k, $k = 1, 2, \ldots, s$, to be distinct eigenvalues of a square matrix $\mathbf{A}$, that is, they are solutions of the polynomial equation

$$\Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}). \tag{3}$$

This polynomial, $\Delta(\lambda)$, is called the characteristic polynomial of the matrix $\mathbf{A}$. The system of linear algebraic equations (2) is sufficient to determine the coefficient functions $b_j(t)$, $j = 0, 1, \ldots, m-1$ only if all eigenvalues of the matrix $\mathbf{A}$ are different[1]. If λ_k is a double root of the characteristic equation $\Delta(\lambda) = 0$ then need to add to Eqs. (2) additional equation:

$$t\,e^{\lambda_k t} = \frac{d}{d\lambda^p}\left[b_0(t) + b_1(t)\lambda + \cdots + b_{m-1}(t)\lambda^{m-1}\right]\Big|_{\lambda=\lambda_k}. \tag{4}$$

Example 1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Its characteristic polynomial is $\Delta(\lambda) = (\lambda+2)^2(\lambda-1)$ has one double root $\lambda_2 = -2$ and a simple root $\lambda_1 = 1$. From Eq. (2) follows that the coefficient functions $b_j(t)$, $j = 0, 1, 2$

$$\begin{aligned} e^t &= b_0(t) + b_1(t) + b_2(t) \\ e^{-2t} &= b_0(t) - 2b_1(t) + 4b_2(t) \\ t\,e^{-2t} &= b_1(t) - 4b_2(t) \end{aligned}$$

From the first equation we find $b_0(t)$ to be

$$b_0(t) = e^t - b_1(t) - b_2(t).$$

Plugging this expression into the second equation leads

$$e^{-2t} = \left(e^t - b_1(t) - b_2(t)\right) - 2b_1(t) + 4b_2(t)$$

or

$$\frac{1}{3}\,e^{-2t} - \frac{1}{3}\,e^t = -b_1(t) + b_2(t).$$

We add this equation with $t\,e^{-2t} = b_1(t) + b_2(t)$ to obtain

$$b_2(t) = \frac{1}{9}\,e^t - \frac{1}{9}\,e^{-2t} - \frac{1}{3}\,t\,e^{-2t}.$$

Now we can determine the other coefficient functions:

$$\begin{aligned} b_1(t) &= \frac{4}{9}\,e^t - \frac{4}{9}\,e^{-2t} - \frac{1}{3}\,t\,e^{-2t} \\ b_0(t) &= \frac{4}{9}\,e^t + \frac{5}{9}\,e^{-2t} + \frac{2}{3}\,t\,e^{-2t} \end{aligned}$$

With this in hand, we obtain the exponential function

$$\begin{aligned} e^{\mathbf{A}t} &= \left(\frac{4}{9}\,e^t + \frac{5}{9}\,e^{-2t} + \frac{2}{3}\,t\,e^{-2t}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad + \left(\frac{4}{9}\,e^t - \frac{4}{9}\,e^{-2t} - \frac{1}{3}\,t\,e^{-2t}\right)\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \\ &\quad + \left(\frac{1}{9}\,e^t - \frac{1}{9}\,e^{-2t} - \frac{1}{3}\,t\,e^{-2t}\right)\begin{bmatrix} 1 & -3 & -3 \\ 3 & 7 & 3 \\ -3 & -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & e^t - e^{-2t} & e^t - e^{-2t} \\ -e^{-t} + e^{-2t} & -e^t + 2\,e^{-2t} & -e^t + e^{-2t} \\ e^t - e^{-2t} & e^t - e^{-2t} & e^t \end{bmatrix}, \end{aligned}$$

because

$$\begin{bmatrix} 1 & -3 & -3 \\ 3 & 7 & 3 \\ -3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}^2 = \mathbf{A}^2.$$

□

[1] This is not always the case. However, here we deal only with matrices with distinct eigenvalues.