Chapter 2: Urban Services

For All Practical Purposes

Mathematical Literacy in Today's World, 9th ed.

Section 2.1 Hamiltonian Circuits

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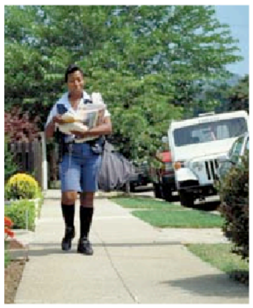
Chapter 2: Business Efficiency
Introduction

- Visiting Vertices
  - In some graph theory problems, it is only necessary to visit specific locations (using the travel routes, or streets available).
  - Problem: Find an efficient route along distinct edges of a graph that visits each vertex only once in a simple circuit.



Applications:

- Salesman visiting particular cities
- Delivering mail to drop-off boxes
- Route taken by a snowplow
- Pharmaceutical representative visiting doctors

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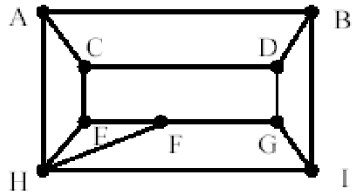
Visiting Vertices



Question: When is it possible to find a route along distinct edges of a graph that visit **each vertex once and only once**?

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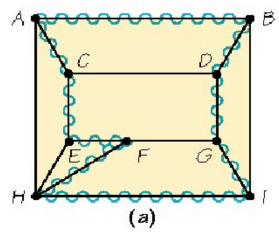
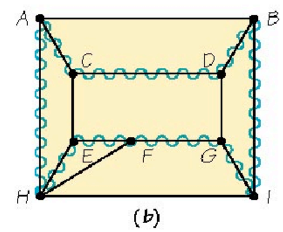
What is a vertex? Think of an application

- Hamiltonian Circuit
  - A tour that starts and ends at the same vertex (circuit definition).
  - Visits each vertex once. (Vertices cannot be reused or revisited.)
  - Circuits can start at any location.
  - Use wiggly edges to show the circuit.
  - Notice a Hamiltonian Circuit uses exactly two edges at each vertex.



Starting at vertex A, the tour can be written as ABDGIHFECA, or starting at E, it would be EFHIGDBACE.



A different circuit visiting each vertex once and only once would be CDBIGFEHAC (starting at vertex C).

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Hamiltonian vs. Euler Circuits

- Similarities
  - Both forbid re-use.
    - Hamiltonian do not reuse vertices.
    - Euler do not reuse edges.

- Differences
  - Hamiltonian is a circuit of vertices.
  - Euler is a circuit of edges.
  - Euler graphs are easy to spot (connectedness and even valence).
  - Hamiltonian circuits are NOT as easy to determine upon inspection.
    - Some certain family of graphs can be known to have or not have Hamiltonian circuits.

Hamiltonian circuit – A tour (showed by wiggly edges) that starts at a vertex of a graph and visits each vertex once and only once, returning to where it started.

Euler circuit – A circuit that traverses each edge of a graph exactly once and starts and stops at the same point.

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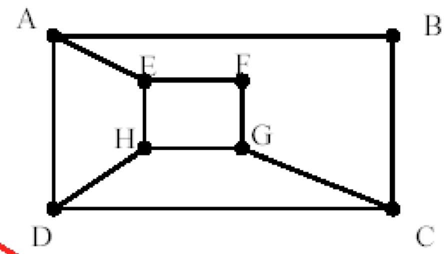
Which path listed forms a Hamiltonian circuit on the graph below?



Not a path

__A. ADCBFGEA

__B. ABCDHGFE

X C. ABCDHGFEA

__D. ABCDHGFEHDA

Not a circuit

Visits vertices D, H twice.

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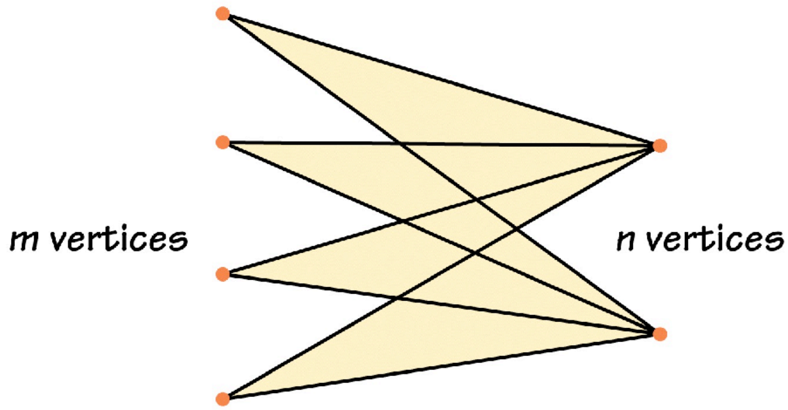
Lack of a Hamiltonian Circuit



There is no easy way to determine whether or not a graph has a Hamiltonian circuit.

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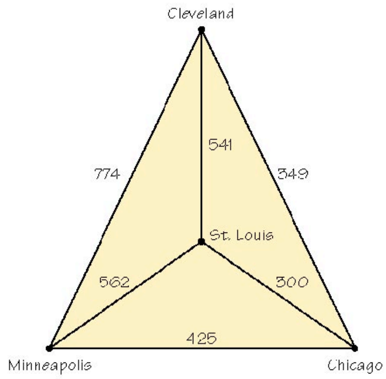
There is illustrates that all graphs do not have a Hamiltonian Circuit. Keep this in mind when doing the essay.

- **Vacation–Planning Problem**
  - Hamiltonian circuit concept is used to find the best route that minimizes the total distance traveled to visit friends in different cities.
    (assume less mileage → less gas → minimizes costs)



Road mileage between four cities

Hamiltonian circuit with weighted edges
- Edges of the graph are given weights, or in this case mileage or distance between cities.
- As you travel from vertex to vertex, add the numbers (mileage in this case).
- Each Hamiltonian circuit will produce a particular sum.

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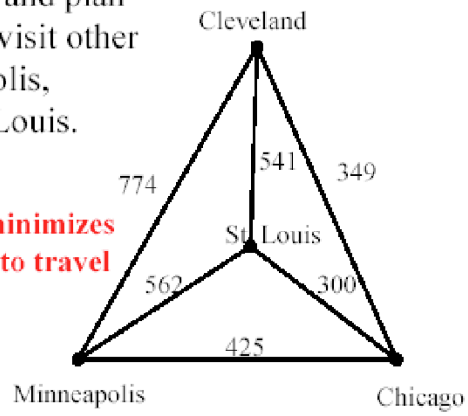
Vacation Planning

We live in Chicago and plan to take a car trip to visit other friends in Minneapolis, Cleveland, and St. Louis.

Design a route that minimizes the distance we have to travel



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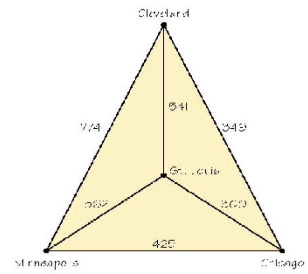
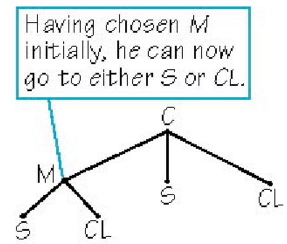


- Minimum-Cost Hamiltonian Circuit
  - A Hamiltonian circuit with the lowest possible sum of the weights of its edges.

■ Algorithm (step-by-step process) for Solving This Problem

1. Generate all possible Hamiltonian tours (starting with Chicago).
2. Add up the distances on the edges of each tour.
3. Choose the tour of minimum distance.



Algorithm – A step-by-step description of how to solve a problem.

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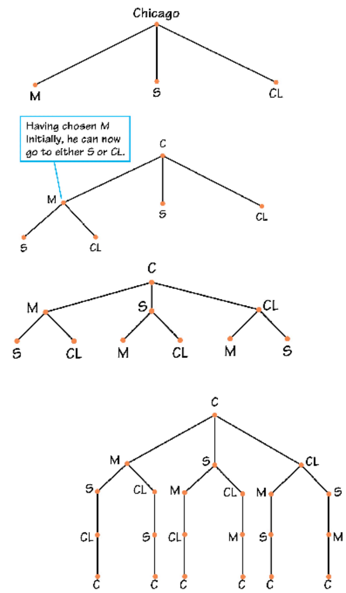
- Method of Trees
  - For the first step of the algorithm, a systematic approach is needed to generate all possible Hamiltonian tours (disregard distances during this step).



- This method begins by selecting a starting vertex, say Chicago, and making a tree-diagram showing the next possible locations.
- At each stage down, there will be one less choice (3, 2, then 1 choice).
- In this example, the method of trees generated six different paths, all starting and ending with Chicago.

However, only three are unique circuits.

Method of trees for vacation-planning problem

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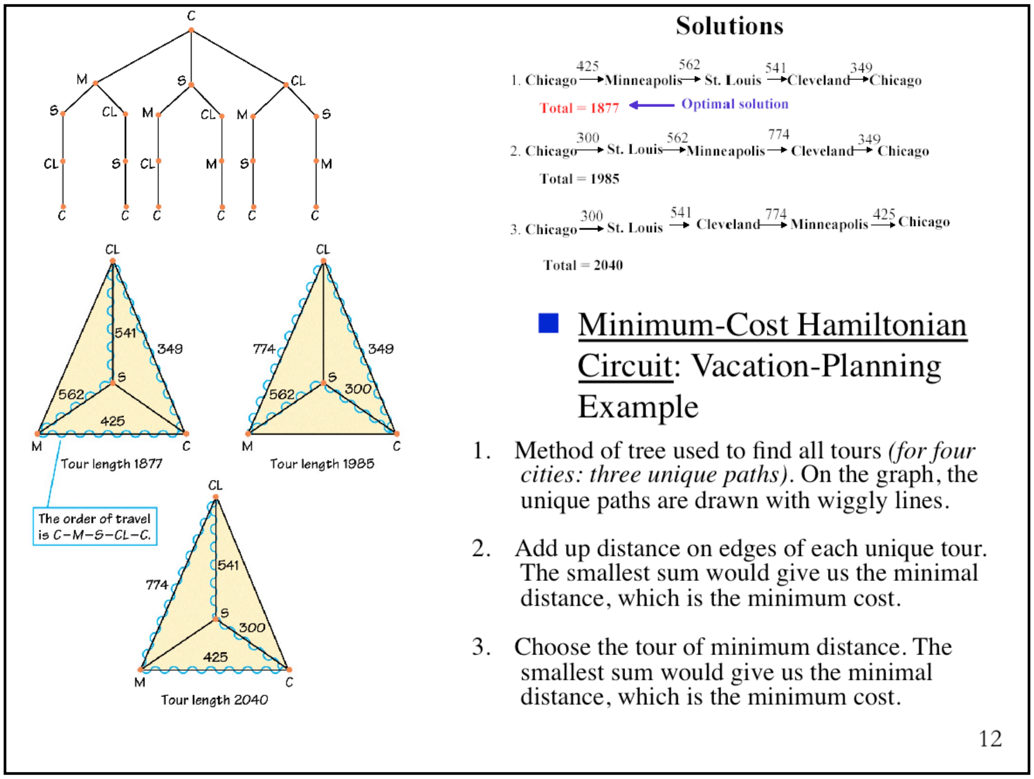
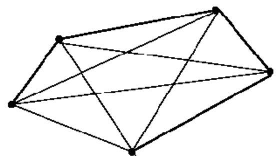


Is this method always possible? Think about how large the graph would be if you had a 25 cities.

Complete Graphs

A graph in which every pair of vertices is joined by an edge.



- Fundamental Principle of Counting

  If there are *a* ways of choosing one thing,
  b ways of choosing a second after the first is chosen,
  c ways of choosing a third after the second is chosen…, and so on…,
  and *z* ways of choosing the last item after the earlier choices,
  then the total number of choice patterns is $a \times b \times c \times \ldots \times z$.

Example: Jack has 9 shirts and 4 pairs of pants. He can wear $9 \times 4 = 36$ shirt-pant outfits.

- Principle of Counting for Hamiltonian Circuits
  - For a complete graph of n vertices, there are $(n - 1)!$ possible routes.
  - Half of these routes are repeats, the result is:

Possible unique Hamiltonian circuits are

$$(n - 1)! / 2$$

Fundamental Principle of Counting for a Complete Graph

If a complete graph has n vertices, then there will be (n-1)!/2 Hamiltonian circuits

$$(n-1)! = (n-1)\times(n-2)\times\cdots\times 3\times 2\times 1$$

which is read "factorial of (n-1)".

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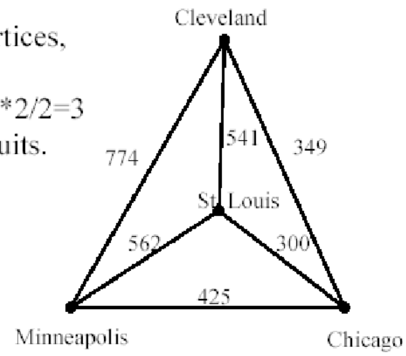
Review our previous example

We have four vertices, thus we have (4-1)!/2=3!/2=3*2/2=3 Hamiltonian circuits.



Suppose you have 25 cities then there would be 24!/2 which is approximately 3 x 10^{23} Hamiltonian circuits. If you had a super computer generating 1 million circuits per second this would take 10 billion years to generate them all. (**Brute Force**)

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