SELF-MAPS OF $\mathbb{P}^2$ WITH INVARIANT ELLIPTIC CURVES

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ABSTRACT. We discuss the geometric and dynamical properties of the holomorphic self-maps of $\mathbb{P}^2$ that leave invariant an elliptic plane curve.

1. INTRODUCTION.

Given an elliptic plane curve Q, we consider the problem of constructing holomorphic self-maps f of $\mathbb{P}^2$ that leave Q invariant. In section 2, we state the criterion for a self-map of Q to extend to $\mathbb{P}^2$. We look in section 3 at the singular points of Q. In contrast with the smooth case, most singular elliptic curves do not admit non-trivial self-maps. The obstructions given by the singular points of Q are discussed in section 3. We define two invariants, in terms of Weierstrass' σ and ζ functions, and state an invariance criterion for the elliptic plane curves with ordinary singularities.

In section 4, we prove that do not exist self-maps of $\mathbb{P}^2$, for which Q is critical and invariant.

We prove in section 5 that the backward orbit of any point of Q is dense in the Julia set of f.

In section 6 we discuss the case when Q is a smooth cubic. The classic tangent process on Q provides examples of self-maps that leave Q invariant. If we require f to leave invariant a line of lines, Q must be isomorphic to the Fermat cubic. We also discuss in this section the case when f has minimal degree 2.

When an elliptic plane curve has enough symmetries, the invariants associated to its singular points can be calculated easily. The simplest case is the dual of a smooth cubic (section 7). In section 8, we consider special families of elliptic quartics with two singular points. Computer-generated pictures illustrate tangent processes on such curves.

2. PRELIMINARIES.

Let $\mathbb{C}_d$ denote the space of homogeneous polynomials of degree d in three variables. A rational self-map f of $\mathbb{P}^2$, of algebraic degree $d(f) = d$ is given by three polynomials p_0, p_1, p_2 in $\mathbb{C}_d$ with no common divisors, according to the formula $f[x_0, x_1, x_2] = [p_0(x_0, x_1, x_2), p_1(x_0, x_1, x_2), p_2(x_0, x_1, x_2)]$. Let $I(f)$ denote the set of indeterminacy of f, formed by the common zeroes in $\mathbb{P}^2$ of the polynomials p_j.

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When $I(f)=\emptyset$, f is regular (holomorphic). We denote by $\deg(\phi)$ the topological degree of a map ϕ. When $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ is regular, $\deg(f)=d(f)^2$.

Every effective divisor D on $\mathbb{P}^2$ of degree $\deg(D)=e$ is given by an equation $(p=0)$, with $0\neq p\in\mathbb{C}_e$ determined by D up to a multiplicative constant. We denote by $\simeq$ the linear equivalence of divisors, and also the isomorphism of line bundles. Two divisors on $\mathbb{P}^2$ are linearly equivalent iff they have the same degree. The pull-back of a divisor D through a map f is denoted by f^*D. Given a rational self-map f of $\mathbb{P}^2$ and a divisor $D=(p=0)$ on $\mathbb{P}^2$ whose support does not contain the image of f, $f^*D=(p(f)=0)$, with $\deg(f^*D)=d(f)\deg(D)$. Let $\mathcal{O}_{\mathbb{P}^2}(d)$ denote the line bundle on $\mathbb{P}^2$ whose global holomorphic sections vanish on effective divisors of degree d. Its restriction to a plane curve Q is denoted $\mathcal{O}_Q(d)$. (By "curve", we mean a one-dimensional irreducible variety.)

Remark 2.1. Let f be a non-constant rational self-map of $\mathbb{P}^2$. If Q is plane curve with $I(f)\cap Q=\emptyset$, then $C:=f(Q)$ is a curve. If $Q\xrightarrow{g}C$ denotes the restriction of f to Q, then $g^*\mathcal{O}_C(1)\simeq\mathcal{O}_Q(d(f))$, hence $\frac{\deg(g)}{d(f)}=\frac{\deg(Q)}{\deg(C)}$.

Proof. Since $I(f)\cap Q=\emptyset$, Q is not contracted by f. Let $C\xrightarrow{i}\mathbb{P}^2$ be the embedding map. By Bertini, the pull-back f^*l of the generic line l in $\mathbb{P}^2$ meets Q transversely, and its trace on Q equals $(ig)^*l$. We get $g^*\mathcal{O}_C(1)\simeq(ig)^*\mathcal{O}_{\mathbb{P}^2}(1)\simeq\mathcal{O}_Q(d(f))$. $\square$

Definition 2.2. A plane curve Q is *invariant* for a rational self-map f of $\mathbb{P}^2$ iff $I(f)\cap Q=\emptyset$ and $f(Q)=Q$. Given a regular self-map g of Q, a regular (resp. rational) *extension* of g to $\mathbb{P}^2$ is a regular (resp. rational) self-map f of $\mathbb{P}^2$ that leaves Q invariant and satisfies $f_{|Q}=g$.

Given a plane curve Q, the degree d divisors on $\mathbb{P}^2$ cut out on Q a complete linear system. This yields a criterion for a self-map of Q to extend to $\mathbb{P}^2$.

Proposition 2.3. *Given a plane curve Q and an integer $d\geq\deg(Q)$, a self-map $Q\xrightarrow{g}Q$ has regular extensions $\mathbb{P}^2\xrightarrow{f}\mathbb{P}^2$ with $d(f)=d$ iff $g^*\mathcal{O}_Q(1)\simeq\mathcal{O}_Q(d)$.*

Proof. Write $Q=(q=0)$, with $q\in\mathbb{C}_e$, $e=\deg(Q)$. Let $i=[s_0,s_1,s_2]$ denote the embedding of Q in $\mathbb{P}^2$, where $s_j\in\Gamma(Q,\mathcal{O}_Q(1))$ are global sections in $\mathcal{O}_Q(1)$. Then $ig=[g^*s_0,g^*s_1,g^*s_2]$, $g^*s_j\in\Gamma(Q,g^*\mathcal{O}_Q(1))$. The map $\mathbb{C}_d\xrightarrow{\beta}\Gamma(Q,\mathcal{O}_Q(d))$, $\beta(p)=p(s_0,s_1,s_2)$ is an epimorphism. For all $c\in Q$, $\beta(p)(c)=0$ iff $p(c)=0$. Consequently, $p\in\ker(\beta)$ iff q divides p.

If $g^*\mathcal{O}_Q(1)\simeq\mathcal{O}_Q(d)$, there exist $p_j\in\mathbb{C}_d$ with $g^*s_j=\beta(p_j)$. For all $c\in Q$, $p_j(c)=0$ iff $s_j(g(c))=0$. Since s_0, s_1, s_2 have no common zeros, p_0, p_1, p_2 have no common zeros on Q. Since p_0, p_1, p_2 and q have no common zeros, for generic $r_j\in\mathbb{C}_{d-e}$, $f:=[p_0+qr_0,p_1+qr_1,p_2+qr_2]$ is a regular self-map of $\mathbb{P}^2$. We have $ig=[g^*s_0,g^*s_1,g^*s_2]=[\beta(p_0),\beta(p_1),\beta(p_2)]=[\beta(p_0+qr_0),\beta(p_1+qr_1),\beta(p_2+qr_2)]=[p_0+qr_0,p_1+qr_1,p_2+qr_2]\circ[s_0,s_1,s_2]=fi$.

If $\mathbb{P}^2\xrightarrow{f}\mathbb{P}^2$ satisfies $f_{|Q}=g$, then, by Remark 2.1, $g^*\mathcal{O}_Q(1)\simeq\mathcal{O}_Q(d(f))$. $\square$

Remark 2.4. Given any integer $d>0$, $Q\xrightarrow{g}Q$ has rational extensions f with $d(f)=d$ iff $g^*\mathcal{O}_Q(1)\simeq\mathcal{O}_Q(d)$. Assuming this, let $e=\deg(Q)$. When $d<e$, g has a unique rational extension f with $d(f)=d$. When $d\geq e$, the regular extensions f of g with $d(f)=d$ form a Zariski open subset of $\mathbb{C}^{3N}$, with $N=\binom{d-e+2}{2}$.

Proof. The first statement follows from the proof of Proposition 2.3. When $d < e$, the evaluation map β is an isomorphism. When $d \geq e$, $\dim(\ker(\beta)) = N$. Now, let $f = [p_0, p_1, p_2]$ and $\tilde{f} = [\tilde{p}_0, \tilde{p}_1, \tilde{p}_2]$ be rational extensions of g with $d(f) = d = d(\tilde{f})$. The map $Q \xrightarrow{k} \mathbb{P}^1$, $k = \frac{\tilde{p}_j}{p_j}$, is independent of j, hence it does not have zeros or poles, i.e. it is constant. We may assume $k = 1$, and then $\tilde{p}_j - p_j \in \ker(\beta)$. □

Remark 2.5. Given a plane curve Q, an integer $d > \deg(Q)$, a map $Q \xrightarrow{g} Q$ with $g^*\mathcal{O}_Q(1) \simeq d\mathcal{O}_Q(1)$, and a point $a \in \mathbb{P}^2 \setminus Q$, there exist regular extensions of g to $\mathbb{P}^2$ for which a is an attracting fixed point.

Proof. Write $Q = (q = 0)$, with $q \in \mathbb{C}_e$. We may assume that $a = [1, 0, 0]$ and $q(1, 0, 0) = 1$. Fix a rational extension of g, $\tilde{f} = [\tilde{p}_0, \tilde{p}_1, \tilde{p}_2]$, with $d(\tilde{f}) = d$. Take $m \in \mathbb{C}_1$ with $m(1, 0, 0) = 0$ so that p_1 and p_2 have no common factors, where $p_j = \tilde{p}_j - \tilde{p}_j(1, 0, 0)x_0^{d-e}q - m^{d-e}q$. Let $\mathcal{B} \subset \mathbb{P}^1$ denote the finite set of common zeros of p_1 and p_2. Note that $a \in \mathcal{B}$. Take $l \in \mathbb{C}_1$ so that $l(1, 0, 0) = 1$ and $l(b) \neq 0$ for all $b \in \mathcal{B}$. For $k \in \mathbb{C}$, put $p_0 = \tilde{p}_0 - kl^{d-e}q$, and define $f_k = [p_0, p_1, p_2]$. If $b \in I(f_k)$, then $b \in \mathcal{B}$, $q(b) \neq 0$ and $k = \tilde{p}_0(b)/(l^{d-e}(b)q(b))$. Therefore, for all but finitely many values of k, f_k is regular. Clearly, $f_k(a) = a$. For large k, a is an attracting fixed point of f_k. □

A curve Q with self-maps of degree greater than 1 must be rational or elliptic. Given a rational plane curve Q, it is easy to find regular self-maps f of $\mathbb{P}^2$ for which Q is invariant and critical. In this paper we discuss the case when Q is elliptic.

Fix a normalization map $C \xrightarrow{\nu} Q$, and a group structure $\mathbb{C} \xrightarrow{[\cdot]} (C, +, 0)$. Given $Q \xrightarrow{g} Q$, let $C \xrightarrow{h} C$ be its lifting through ν, $\nu h = g\nu$. There exist m, $n \in \mathbb{C}$ so that $h[t] = [mt - n]$ for all $t \in \mathbb{C}$. The *multiplier* $m(g) := m$ does not depend on ν or $[\cdot]$, and $\deg(g) = |m|^2$. Let $\mathbb{Z}(C)$ denote the ring formed by the multipliers of the self-maps of C, and $\mathbb{U}(C)$ the group of units in $\mathbb{Z}(C)$. Given a rational self-map f of $\mathbb{P}^2$ that leaves Q invariant, $m_Q(f)$ denotes the multiplier of its restriction to Q. Given $0 \neq m \in \mathbb{Z}(C)$, we wish to construct self-maps of $\mathbb{P}^2$ that leave Q invariant, with $m_Q(f) = m$. To do this, we must find $n \in \mathbb{C}$ so that the self-map $[t] \mapsto [mt - n]$ of C induces through ν a regular self-map g of Q with $g^*\mathcal{O}_Q(1) \simeq \mathcal{O}_Q(|m|^2)$.

Definition 2.6. Given $0 \neq m \in \mathbb{Z}(C)$, $R_Q(m)$ denotes the set of points $[n] \in C$ with the property that the self-map $[t] \mapsto [mt - n]$ of C induces through ν a self-map of Q that admits rational extensions to $\mathbb{P}^2$. Let $r_Q(m)$ be the cardinality of $R_Q(m)$.

With a choice of normalization ν and group structure $[\cdot]$, $R_Q(m)$ is identified with the set of self-map of Q with multiplier m that extend rationally to $\mathbb{P}^2$.

3. Singularities.

3.1. **Multiplicities.** Denote by $m_a(A)$ the multiplicity, and by $T_a(A)$ the tangent of an irreducible curve germ $(A, a) \subset (\mathbb{C}^2, a)$.

Lemma 3.1. *Let $(A, a) \subset (\mathbb{C}^2, a)$ and $(B, b) \subset (\mathbb{C}^2, b)$ be irreducible curve germs, with normalizations $(\tilde{A}, \tilde{a}) \xrightarrow{\nu_A} (A, a)$ and $(\tilde{B}, \tilde{b}) \xrightarrow{\nu_B} (B, b)$. Let $(\mathbb{C}^2, a) \xrightarrow{f} (\mathbb{C}^2, b)$ be a finite map germ with $f(A, a) = (B, b)$. Denote by $(A, a) \xrightarrow{g} (B, b)$ the restriction of f, and by $(\tilde{A}, \tilde{a}) \xrightarrow{\tilde{g}} (\tilde{B}, \tilde{b})$ the lifting of g through ν_A and ν_B.*

1. *If $d\tilde{g}(\tilde{a}) \neq 0$ then $m_a(A) \leq m_b(B)$, and $m_a(A) = m_b(B)$ iff $df(a)_{|T_a(A)} \neq 0$.*

2. *If* $d\tilde{g}(\tilde{a}) = 0$ *and* $m_a(A) \le m_b(B)$, *then* $df(a)_{|T_a(A)} = 0$.

Proof. We explicite the Puisseux series of (A, a) and (B, b), and then identify the coefficients in the Taylor series at $\tilde{a}$ of $\nu_B \tilde{g} = f\nu_A$.

Choose local coordinates (x, y) near $a = (0,0) \in \mathbb{C}^2$, (u, v) near $b = (0,0) \in \mathbb{C}^2$, s near $\tilde{a} = 0 \in \tilde{A}$, and t near $\tilde{b} = 0 \in \tilde{B}$ so that $T_a(A) = (y = 0)$, $T_b(B) = (v = 0)$, $\nu_A(s) = (s^m, O_{m+1}(s))$, and $\nu_B(t) = (t^p, O_{p+1}(t))$, with $m = m_a(A)$, $p = m_b(B)$. Here, $O_{m+1}(\cdot)$ denotes a holomorphic function involving terms of degree at least $m+1$. Write $f(x, y) = (u, v) = (\alpha x + \beta y + O_2(x, y), \gamma x + \delta y + O_2(x, y))$, and $\tilde{g}(s) = t = ks + O_2(s)$. Calculate $\nu_B \tilde{g}(s) = (k^p s^p + O_{p+1}(s), O_{p+1}(s))$, and $f\nu_A(s) = (\alpha s^m + O_{m+1}(s), \gamma s^m + O_{m+1}(s))$. Therefore, $\alpha s^m + O_{m+1}(s) = k^p s^p + O_{p+1}(s)$, and $\gamma s^m + O_{m+1}(s) = O_{p+1}(s)$.

Assume first that $d\tilde{g}(\tilde{a}) \neq 0$, i.e. $k \neq 0$. Then $p \ge m$, or else $k^p s^p = O_{p+1}(s)$. If $p = m$, then $\alpha = k^p$ and $\gamma = 0$, hence $df(0)(\partial_x) = \alpha\partial_u + \gamma\partial_v = k^m \partial_u$. If $p > m$, then $\alpha = \gamma = 0$, hence $df(0)(\partial_x) = 0$.

If $k = 0$ and $m \le p$, then $\alpha = \gamma = 0$, hence $df(0)(\partial_x) = 0$. □

Lemma 3.2. *With the notations of Lemma 3.1, assume that* $(A, a) = (B, b)$.

1. *If* $d\tilde{g}(\tilde{a})$ *acts on* $T_{\tilde{a}}(\tilde{A})$ *as multiplication by* $k \in \mathbb{C}$, *then* $df(a)$ *acts on* $T_a(A)$ *as multiplication by* $k^{m_a(A)}$.
2. *If* $f^*(A, a) > (A, a)$ *and* $m_a(A) > 1$, *then* $d\tilde{g}(\tilde{a}) = 0$ *and* $df^2(a) = 0$.

Proof. The first statement follows from the proof of Lemma 3.1. To show the second statement, choose coordinates (x, y) near $a \in \mathbb{C}^2$, and s near $\tilde{a} = 0 \in \tilde{A}$, so that $\nu_A(s) = (s^m, s^n + O_{n+1}(s))$, with $1 < m < n$ and $n/m \notin \mathbb{Z}$. Write $\tilde{g}(s) = ks + O_2(s)$, and $f(x, y) = (u, v)$, with $u = \alpha x + \beta y + O_2(x, y)$, and $v = \gamma x + \delta y + O_2(x, y)$.

Since $(A, 0)$ is defined by $y^m + O_{m+1}(x, y)$, $f^*(A, 0)$ is defined by $v^m + O_{m+1}(u, v)$. Since $f^*(A, 0) > (A, 0)$, we get $v^m + O_{m+1}(u, v) = (y^m + O_{m+1}(x, y))O_1(x, y)$. Therefore, $v^m = O_{m+1}(x, y)$, hence $v = O_2(x, y)$, i.e. $\gamma = \delta = 0$.

Since $\nu_A \tilde{g}(s) = (k^m s^m + O_{m+1}(s), k^n s^n + O_{n+1}(s))$, and $f\nu_A(s) = (u(s^m, s^n) + O_{n+1}(s), v(s^m, s^n) + O_{n+1}(s))$, we get $v(s^m, s^n) = k^n s^n + O_{n+1}(s)$. Assume $k \neq 0$, and write $v(x, y) = w(x) + yO_1(x, y)$. Then $w(s^m) + O_{m+n}(s) = k^n s^n + O_{n+1}(s)$, hence $w(s^m) = k^n s^n + O_{n+1}(s)$. Clearly, $w \not\equiv 0$. Write $w(x) = hx^p + O_{p+1}(x)$, with $h \neq 0$ and $p \ge 2$. If $mp < n$, then $h = 0$. If $mp = n$, then $n/m \in \mathbb{Z}$. If $mp > n$, then $k = 0$. In any case, we get a contradiction. Therefore, $k = 0$.

By (1), $\alpha = k^m$, hence $\alpha = 0$. It follows that $df^2(a) = (df(a))^2 = 0$. □

Lemma 3.3. *Let* $(Q, 0) \subset (\mathbb{C}^2, 0)$ *be a curve germ with branches* $(A_i, 0)$ *normalized by* $(\tilde{A}_i, \tilde{a}_i) \xrightarrow{\nu_i} (A_i, 0)$. *Let* $(\mathbb{C}^2, 0) \xrightarrow{f} (\mathbb{C}^2, 0)$ *be a finite map germ that satisfies* $f(Q, 0) \subset (Q, 0)$, *inducing* $(A_i, 0) \xrightarrow{g_i} (A_{j_i}, 0)$. *Let* $(\tilde{A}_i, \tilde{a}_i) \xrightarrow{\tilde{g}_i} (\tilde{A}_{j_i}, \tilde{a}_{j_i})$ *be the lifting of* g_i *through* ν_i, ν_{j_i}. *If* $d\tilde{g}_i(\tilde{a}_i) \neq 0$ *for all* i, *then* $df(0)_{|T_0(A_i)} \neq 0$ *for all* i.

Proof. Since every $(A_i, 0)$ is pre-periodic for f, replacing f by an iterate, we may assume that $f^2(A_i) = f(A_i)$ for all i.

When $f(A_i) \neq A_i$, we have $m_0(f(A_i)) = 1 = m_0(A_i)$ and $df(0)_{|T_0(A_i)} \neq 0$. Indeed, in this case $f^*(f(A_i)) \ge f(A_i) + A_i > f(A_i)$, hence, by Lemma 3.2.2, $m_0(f(A_i)) = 1$. By Lemma 3.1.1, $m_0(A_i) \le m_0(f(A_i))$, hence $m_0(A_i) = 1$, and then $df(0)_{|T_0(A_i)} \neq 0$.

When $f(A_i) = A_i$, Lemma 3.1.1 implies that $df(0)_{|T_0(A_i)} \neq 0$. □

3.2. **Torsion.** Let $\mathbb{C} \xrightarrow{[\cdot]} (C,+,0)$ be a smooth elliptic curve. Given a divisor D on C, let $s(D) := \sum_{c\in D} c$ (multiplicities are counted in all such sums and products indexed by divisors). By Jacobi's theorem, $D \simeq 0$ iff $\deg(D) = 0$ and $s(D) = 0$. Given $0 \neq m \in \mathbb{Z}(C)$, let C_m be the kernel of the morphism $[t] \mapsto [mt]$, and $c_m := s(C_m)$. Note that $c_m \in C_m \cap C_2$. When $m \in \mathbb{Z}$ or $|m|^2$ is odd, $c_m = 0$.

Remark 3.4. Let D be a divisor on C with $0 \neq e := \deg(D)$ and $s(D) = 0$. Given $0 \neq m \in \mathbb{Z}(C)$ and $n \in \mathbb{C}$, the map $C \xrightarrow{g} C$, $g[t] = [mt - n]$, satisfies $g^*D \simeq |m|^2 D$ iff $e[\overline{m}n] = ec_m$. In this case, $g(C_{e|m|^2}) \subset C_{e|m|^2}$.

Proof. We calculate $s(g^*D) = e([\overline{m}n] + c_m)$, hence $s(g^*D) = 0$ iff $e[\overline{m}n] = ec_m$. Such g satisfies $e|m|^2[n] = 0$, hence $g(C_{e|m|^2}) \subset C_{e|m|^2}$. □

Given a divisor D on C, write $D \simeq_\tau 0$ iff there exists a positive integer k so that $kD \simeq 0$. Let $\tau(D)$ be the smallest such k. When $D \not\simeq_\tau 0$, let $\tau(D) := +\infty$. Given $a, b \in C$, write $a \simeq_\tau b$ iff $(a) - (b) \simeq_\tau 0$. Let $\tau(a,b) := \tau((a) - (b))$.

Remark 3.5. Let $C \xrightarrow{h} C$ be a self-map of a smooth elliptic curve, $0 \neq m = m(h)$. Given any points a and b in C, $\frac{\tau(a,b)}{|m|^2} \leq \tau(h(a), h(b)) \leq \tau(a,b)$.
If $|m| > 1$ and a is pre-periodic for h, then $a \simeq_\tau b$ iff b is pre-periodic for h.

Proof. We may assume that h is a morphism of $(C,+,0)$. If $k(a-b) = 0$, then $k(h(a) - h(b)) = h(0) = 0$. If $k(h(a) - h(b)) = 0$, then $k|m|^2(a-b) = 0$.

Let a be pre-periodic for h. Replacing h by an iterate, we may assume that $h^2(a) = h(a)$, hence $|m^2 - m|^2 a = 0$. When $|m| > 1$, we get $a \simeq_\tau 0$.

If $b \simeq_\tau a$, then $b \simeq_\tau 0$, i.e. $kb = 0$ for some $0 < k \in \mathbb{Z}$. The h-orbit of b is contained in the finite set of points $c \in C$ with $kc = 0$, hence b is pre-periodic. □

Most elliptic plane curves do not admit self-maps of degree greater than 1:

Proposition 3.6. *Let Q be an elliptic plane curve with normalization $C \xrightarrow{\nu} Q$. Given $Q \xrightarrow{g} Q$ with $\deg(g) > 1$, let $C \xrightarrow{h} C$ be the lifting of g through ν. Then:*

1. *The singular branches of Q are pre-periodic for h. If a and b are singular branches of Q, then $a \simeq_\tau b$.*
2. *If a and b are branches of Q with $\nu(a) = \nu(b)$, then $a \simeq_\tau b$.*

Proof. Let $S \subset C$ be the finite set of singular branches of Q. By Lemma 3.1.(1), $h(S) \subset S$. Remark 3.5 finishes the proof of (1).

To prove (2), let $q = \nu(a) = \nu(b)$. If q is pre-periodic for g, then a and b are pre-periodic for h, hence $a \simeq_\tau b$. If q is not pre-periodic for g, replacing g by an iterate we may assume that $(Q, g(q))$ is irreducible. Then $h(a) = h(b)$, hence $a \simeq_\tau b$. □

3.3. **Ordinary singularities.** Let Q be an elliptic plane curve, with normalization $C \xrightarrow{\nu} Q$ and group structure $\mathbb{C} \xrightarrow{[\cdot]} C$. A germ (Q,q) is an *ordinary* singularity iff $m_q(Q) = 2$ and the proper transform of (Q,q) through the blow-up of $\mathbb{P}^2$ at q is smooth. In suitable local coordinates near q, an ordinary singularity (Q,q) is either the cusp ($y^2 = x^3$) or the node ($xy = 0$).

Lemma 3.7. *Given $C \xrightarrow{\phi} \mathbb{P}^1$, let $\psi := \phi\nu^{-1}$ be defined on the smooth locus of Q. If ψ is regular on Q, the following conditions are satisfied:*

1. *If $[a]$ is a singular branch of Q, then $[a]$ is critical for ϕ.*

2. *If $\nu[a] = \nu[b]$, then $\phi[a] = \phi[b]$.*

When Q has ordinary singularities, these conditions imply that ψ is regular on Q.

Proof. Assume ψ is regular on Q, so that $\psi\nu = \phi$. If $\nu[a] = \nu[b]$, clearly $\phi[a] = \phi[b]$. The singular branches of Q are critical for ν, hence they are critical for ϕ.

Assume that C has ordinary singularities. Given a node $\nu[a] = \nu[b]$, let ψ_a, ψ_b be the restrictions of ψ to the branches $[a]$, $[b]$ (respectively). Clearly, ψ_a and ψ_b are regular. If $\phi[a] = \phi[b]$, then $\psi_a(\nu[a]) = \psi_b(\nu[a])$, hence ψ is regular at $\nu[a]$. Given a cusp $\nu[a]$, in suitable local coordinates t near $[a]$ and (x, y) near $\nu[a]$, $\nu(t) = (t^2, t^3)$. If $[a]$ is critical for ϕ, then $\phi(t)$ is a holomorphic function of t^2 and t^3. □

Lemma 3.8. *Given $C \xrightarrow{h} C$, let $g := \nu h \nu^{-1}$ be defined on the smooth locus of Q. If g is regular on Q, the following conditions are satisfied:*

1. *If $[a]$ is a singular branch of Q, then $h[a]$ is also singular.*
2. *If $\nu[a] = \nu[b]$, then $\nu h[a] = \nu h[b]$.*

When Q has ordinary singularities, these conditions imply that g is regular on Q.

Proof. Assume g is regular on Q, so that $g\nu = \nu h$. If $\nu[a] = \nu[b]$, clearly $\nu h[a] = \nu h[b]$. If $[a]$ is a singular branch, Lemma 3.1 (1) implies that $h[a]$ is also singular.

Assume that C has ordinary singularities. Given a node $\nu[a] = \nu[b]$, let g_a, g_b be the restrictions of g to the branches $[a]$, $[b]$ (respectively). Clearly, g_a and g_b are regular. If $\nu h[a] = \nu h[b]$, then $g_a(\nu[a]) = g_b(\nu[a])$, hence g is regular at $\nu[a]$. If $h(\nu[a])$ is a cusp, choose, as before, local coordinates t near $[a]$ and (x, y) near $\nu[a]$, so that $\nu(t) = (t^2, t^3)$. Since $[a]$ is critical for νh, $\nu h(t)$ is a holomorphic function of t^2 and t^3, hence g is regular at $\nu[a]$. □

3.4. **Invariants at singular points.** Recall the definition and basic properties of the Weierstrass functions σ, ζ and $\mathcal{P}$. Let Ω be a lattice in $\mathbb{C}$, with associated elliptic curve $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega = (C, +, 0)$. Fix a positively oriented basis (λ_1, λ_2) in Ω. Given two lattice points $\omega_i = a_i\lambda_1 + b_i\lambda_2 \in \Omega$, $\det(\omega_1, \omega_2) := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The odd entire function $\sigma(t) := t \prod_{0 \neq \omega \in \Omega} \left(1 - \frac{t}{\omega}\right) \exp\left(\frac{t}{\omega} + \frac{1}{2}\left(\frac{t}{\omega}\right)^2\right)$ has the translation property $\frac{\sigma(t+\omega)}{\sigma(t)} = \epsilon(\omega) \exp(\frac{1}{2}\eta(\omega)\omega) \exp(\eta(\omega)t)$, for all $t \in \mathbb{C}$ and $\omega \in \Omega$. Here, $\epsilon(\omega) = \begin{cases} 1, & \text{when } \omega \in 2\Omega \\ -1, & \text{when } \omega \notin 2\Omega \end{cases}$, and $\Omega \xrightarrow{\eta} \mathbb{C}$ is a group morphism that satisfies the Legendre relation $\begin{vmatrix} \eta(\omega_1) & \omega_1 \\ \eta(\omega_2) & \omega_2 \end{vmatrix} = 2\pi \mathrm{i} \det(\omega_1, \omega_2)$.

The odd meromorphic function $\zeta := (\log \sigma)' = \frac{1}{t} + \sum_{0 \neq \omega \in \Omega} \left(\frac{1}{t-\omega} + \frac{1}{\omega} + \frac{t}{\omega^2}\right)$ has the translation property $\zeta(t + \omega) = \zeta(t) + \eta(\omega)$, for all $t \in \mathbb{C}$ and $\omega \in \Omega$.

The function $\mathcal{P} := \zeta' = -\frac{1}{t^2} - \sum_{0 \neq \omega \in \Omega} \left(\frac{1}{(t-\omega)^2} - \frac{1}{\omega^2}\right)$ is even and Ω-periodic.

Given a divisor A on $\mathbb{C}$, let $[A] := \sum_{a \in A} ([a])$, and $s(A) := \sum_{a \in A} a$. When $[A] \simeq 0$, the meromorphic function $\Phi_A(t) := \exp(\eta(s(A))t) \prod_{a \in A} \sigma(t - a)$ is Ω-periodic, inducing an elliptic function $C \xrightarrow{\phi_A} \mathbb{P}^1$, with principal divisor $(\phi_A) = [A]$.

Lemma 3.9. *Let D be an effective divisor on $C = \mathbb{C}/\Omega$, with $\deg(D) := e$ and $s(D) = 0$. Given $0 \neq m \in \mathbb{Z}$ and $n \in \mathbb{C}$ with $emn \in \Omega$, consider the self-map*

$C \xrightarrow{h} C$, $h[t] := [mt-n]$, and the meromorphic function $\mathbb{C} \xrightarrow{\Phi} \mathbb{P}^1$, $\Phi(t) := \exp(t\eta(emn)) \prod_{a\in\widehat{D}} \frac{\sigma(mt-n-a)}{\sigma^{m^2}(t-a)}$. Then Φ is Ω-periodic, inducing on C an elliptic function $C \xrightarrow{\phi} \mathbb{P}^1$ with $(\phi) = h^*D - m^2D$.

Proof. Let $\lambda = emn$, and fix $\omega \in \Omega$. Since $m \in \mathbb{Z}$ and $s(\widehat{D}) = 0$, the translation property of σ and the Legendre identity imply that

$$\frac{\Phi(t+\omega)}{\Phi(t)} = \exp(\omega\eta(\lambda)) \prod_{a\in\widehat{D}} \frac{\epsilon(m\omega)\exp(\frac{1}{2}m\omega\eta(m\omega))\exp((mt-n-a)\eta(m\omega))}{\epsilon(m^2\omega)\exp(\frac{1}{2}m^2\omega\eta(\omega))\exp(m^2(t-a)\eta(\omega))} =$$
$$\exp(\omega\eta(\lambda) - \lambda\eta(\omega))\exp((m^2-m)s(\widehat{D})\eta(\omega)) = 1.$$ □

Let Q be an elliptic plane curve of degree e. Choose a normalization $C \xrightarrow{\nu} Q$, and a group structure $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega = (C, +, 0)$ so that $s(\nu^*\mathcal{O}_Q(1)) = 0$. Given a line l in $\mathbb{P}^2$, let $l_C := (i\nu)^*l$, where $Q \xrightarrow{i} \mathbb{P}^2$ is the inclusion map. Given a divisor D on C with $s(D) = 0$, $\widehat{D}$ denotes a divisor on $\mathbb{C}$ with $[\widehat{D}] = D$ and $s(\widehat{D}) = 0$.

Definition 3.10. Given two points a, b in $\mathbb{C}$ and a line l in $\mathbb{P}^2$ not passing through $\nu[a]$ or $\nu[b]$, define

$$\alpha(a,l) := \sum_{t\in\widehat{l_C}} \zeta(a-t) \quad \text{and} \quad \alpha(a,b,l) := \prod_{t\in\widehat{l_C}} \frac{\sigma(a-t)}{\sigma(b-t)}.$$

The translation properties of ζ and σ imply that $\alpha(a,l)$ and $\alpha(a,b,l)$ do not depend on the choice of divisor $\widehat{l_C}$ on $\mathbb{C}$ satisfying $[\widehat{l_C}] = l_C$ and $s(\widehat{l_C}) = 0$.

Remark 3.11. (3.4) The translation properties of α come from those of ζ and σ. Given ω, $\lambda \in \Omega$, $\alpha(a+\omega,l) - \alpha(a,l) = e\eta(\omega)$, $\frac{\alpha(a+\omega,b+\omega,l)}{\alpha(a,b,l)} = \exp(e\eta(\omega)(a-b))$, and $\frac{\alpha(a+\omega,b+\lambda,l)}{\alpha(a,b,l)} = \left(\frac{\epsilon(\omega)}{\epsilon(\lambda)}\right)^e \exp\left(\frac{e}{2}(\eta(\omega)\omega - \eta(\lambda)\lambda)\right)\exp\left(e(\eta(\omega)a - \eta(\lambda)b)\right)$.

Recall that $\check{\mathbb{P}}^2$ denotes the space of lines in $\mathbb{P}^2$. Given $p \neq q$ in $\mathbb{P}^2$, $L(p,q)$ is the line passing through p and q. Given $p \in \mathbb{P}^2$, $\check{p} := \{l \in \check{\mathbb{P}}^2 : p \in l\}$.

Proposition 3.12. *When $\nu[a] = \nu[b]$, $\alpha(a,b,l)$ is independent of $l \in \check{\mathbb{P}}^2 \setminus \nu[a]\check{}$. When $[a]$ is a singular branch of Q, $\alpha(a,l)$ is independent of $l \in \check{\mathbb{P}}^2 \setminus \nu[a]\check{}$.*

Proof. Fix two lines $l_0 \neq l_\infty$ in $\mathbb{P}^2$ not passing through $q := \nu[a]$, and let $p := l_0 \cap l_\infty$, $l_1 := L(p,q)$. Let $C \xrightarrow{\phi} \check{p}$ be the central projection $\phi[x] := L(p, \nu[x])$. Given $l \in \check{p}$, a local study near the intersection of Q with l shows that $\phi^*(l) = l_C$. Pick coordinates in $\check{p} = \mathbb{P}^1$ with $l_0 = 0$ and $l_\infty = \infty$. Let $A := \widehat{(l_0)_C} - \widehat{(l_\infty)_C}$. Clearly, $\deg(A) = 0$ and $s(A) = 0$. Since $(\phi_A) = (l_0)_C - (l_\infty)_C = (\phi)$, there exists $0 \neq k \in \mathbb{C}$ with $\phi = k\phi_A$.

If $\nu[b] = q$, then $\phi[a] = \phi[b] = l_1$, hence $\frac{\alpha(a,b,l_0)}{\alpha(a,b,l_\infty)} = \frac{\phi_A[a]}{\phi_A[b]} = \frac{\phi[a]}{\phi[b]} = 1$.

If $[a]$ is a singular local branch of Q, then $\phi^*(l_1) = (l_1)_C = \nu^* i^* l_1 \geq \nu^* q \geq 2[a]$, i.e. $\phi'[a] = 0$. We get $\alpha(a,l_0) - \alpha(a,l_\infty) = \frac{\phi_A'}{\phi_A}[a] = \frac{\phi'}{\phi}[a] = 0$. □

Definition 3.13. Let $0 \neq m \in \mathbb{Z}$ and $[n] \in C_{em}$.
Given a singular branch $[a]$ of Q so that $[ma-n]$ is also singular, define

$$\alpha_{m,n}(a) := m\alpha(ma-n) - m^2\alpha(a) + \eta(emn).$$

Given two branches $[a]$, $[b]$ of Q with $\nu[a] = \nu[b]$ and $\nu[ma-n] = \nu[mb-n]$, define

$$\alpha_{m,n}(a,b) := \frac{\alpha(ma-n, mb-n)}{\alpha^{m^2}(a,b)} \cdot \exp((a-b)\eta(emn)).$$

Clearly, $\alpha_{m,n}(a)$ and $\alpha_{m,n}(a,b)$ are well-defined, and depend only on $[a]$, $[b]$, $[n]$. Moreover, $\alpha_{m,n}(a,b)\alpha_{m,n}(b,a) = 1$.

Theorem 3.14. *Let Q be an elliptic plane curve of degree e. Choose a normalization $C \xrightarrow{\nu} Q$, and a group structure $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega = (C, +, 0)$ so that $s(\nu^*\mathcal{O}_Q(1)) = 0$. Assume that Q has ordinary singularities. Then, given $0 \neq m \in \mathbb{Z}$, $R_Q(m)$ is formed by the points $[n] \in C_{em}$ with the following properties:*

1. *If $[a]$ is a singular branch of Q, then $[ma-n]$ is singular and $\alpha_{m,[n]}[a] = 0$.*
2. *If $\nu[a] = \nu[b]$, then $\nu[ma-n] = \nu[mb-n]$ and $\alpha_{m,[n]}([a],[b]) = 1$.*

Proof. Given $0 \neq m \in \mathbb{Z}$ and $[n] \in C$, let $h[t] := [mt-n]$, and define $g := \nu h \nu^{-1}$ on the smooth locus of Q. Fix a generic line l in $\mathbb{P}^2$, and let $l_Q := i^*l$, $l_C := \nu^* l_Q$.

If $[n] \in R_Q(m)$, g extends rationally to $\mathbb{P}^2$. By Remark 2.4, $g^*l_Q \simeq m^2 l_Q$. Since $g\nu = \nu h$, $h^*l_C \simeq m^2 l_C$. By Remark 3.4, $[n] \in C_{em}$. By Lemma 3.8, if $[a]$ is a singular then $h[a]$ is singular. If $\nu[a] = \nu[b]$, then $\nu h[a] = \nu h[b]$. Now, there exists $Q \xrightarrow{\psi} \mathbb{P}^1$ with $(\psi) = g^*l_Q - m^2 l_Q$, hence $(\psi\nu) = h^*l_C - m^2 l_C$. By Lemma 3.9 applied to l_C, we may assume that $\psi\nu$ is induced by the Ω-periodic function $\Phi(t) := \exp(t\eta(emn)) \prod_{a \in \widehat{l_C}} \frac{\sigma(mt-n-a)}{\sigma^{m^2}(t-a)}$. By Lemma 3.7, if $[a]$ is a singular, then $\Phi'(a) = 0$, i.e. $\alpha_{m,[n]}[a] = 0$. If $\nu[a] = \nu[b]$, then $\Phi(a) = \Phi(b)$, i.e. $\alpha_{m,[n]}([a],[b]) = 1$.

Assume that $[n] \in C_{em}$ satisfies the two conditions. Since the singularities of Q are ordinary, g is regular, by Lemma 3.8. By Remark 3.4, $h^*l_C \simeq m^2 l_C$. Let $C \xrightarrow{\phi} \mathbb{P}^1$ be induced by Φ, with Φ defined as above. By Lemma 3.7, $\psi := \phi\nu^{-1}$ is regular on Q. Since $(\phi) = h^*l_C - m^2 l_C$, we get $(\psi) = g^*l_Q - m^2 l_Q$, hence $g^*l_Q \simeq m^2 l_Q$. Remark 2.4 finishes the proof. ☐

Remark 3.15. Given any elliptic plane curve Q, $R_Q(m) \subset C_{em}$, and the points of $R_Q(m)$ satisfy the properties stated in Theorem 3.14.

As an application of Theorem 3.14, we have the following.

Corollary 3.16. *With the notations of Theorem 3.14, assume that Q is singular, and that the singularities of Q are ordinary cusps. If $0 \neq k \in \mathbb{Z}$ has the property that $k[a] = 0$ for all cusps $[a]$, then, given $m \in \mathbb{Z}$ with $|m| > 1$, $R_Q(m) \subset R_Q(m+k)$.*

Proof. Fix $[n] \in R_Q(m)$. Given a cusp $[a_0]$, let $a_{j+1} := ma_j - n$, for $j \geq 0$. Since $[a_j]$ is a cusp for all j, $[a_s] = [a_r]$ for some $0 \leq s < r$. Clearly, $a_j = m^j a_0 - \frac{m^j - 1}{m-1} n$.

By Remark 3.5, there exists such k. Let $\omega_j := ka_j$, $\omega := kn$, and $\alpha_j := \frac{k}{e}\alpha(a_j)$. Since $\alpha_{m,n}[a_j] = 0$, we get $\alpha_{j+1} = m\alpha_j - \eta(\omega)$. By Remark 3.4, $k\eta(a_r - a_s) = \alpha_r - \alpha_s = (m^r - m^s)(\alpha_0 - \frac{\eta(\omega)}{m-1})$, hence $\alpha_0 = \eta(\omega_0)$.

We calculate $\frac{k}{e(m+k)}\alpha_{m+k,n}(a_0) = \frac{k}{e}\alpha(a_1 + \omega_0) - (m+k)\alpha_0 + \eta(\omega) = \alpha_1 + k\eta(\omega_0) - (m+k)\eta(\omega_0) + \eta(\omega) = 0$. ☐

4. Invariant critical components.

In this section we prove that, given an elliptic plane curve Q, there do not exist self-maps of $\mathbb{P}^2$ for which Q is critical and invariant.

Theorem 4.1. *Given a plane curve Q, the following are equivalent:*

1. *There exist self-maps of $\mathbb{P}^2$ for which Q is invariant and critical.*
2. *The curve Q is rational.*

Proof. Assume that Q is invariant and critical for a rational self-map f of $\mathbb{P}^2$. By Remark 2.1, Q is rational or elliptic. Assume that Q is elliptic. Fix a normalization map $C \xrightarrow{\nu} Q$, and consider the incidence surface $S := \{(c,l) \in C \times \check{\mathbb{P}}^2 :\ \nu(c) \in l\}$, with canonical projections $S \xrightarrow{p} C$, $p(c,l) = c$, and $S \xrightarrow{\pi} \check{\mathbb{P}}^2$, $\pi(c,l) = l$. Clearly, S is smooth, $S \xrightarrow{p} C$ is a ruled surface, and π is finite, with $\deg(\pi) = \deg(Q) := e$. Fix $r \in \mathbb{P}^2$, and let L be the graph of the projection $C \ni c \mapsto L(r, \nu(c)) \in \check{\mathbb{P}}^2$. For generic r, we have $\pi^* \check{r} = L$, hence $L^2 = e$.

By Lemma 3.1 and Lemma 3.2, Q cannot have singular branches, hence the curve $\check{Q}$ dual to Q has degree $2e$ in $\check{\mathbb{P}}^2$. Let T be the graph of the map dual to ν, $C \ni c \mapsto T_{\nu(c)}c \in \check{\mathbb{P}}^2$. We see that $LT = \deg(\check{Q}) = 2e$.

Given $q \in Q$, the differential $df(q)$ has 1-dimensional kernel X_q, which we identify with the line in $\mathbb{P}^2$ whose tangent space at q is X_q. Let X be the graph of the map $C \ni c \mapsto X_{\nu(c)} \in \check{\mathbb{P}}^2$. Clearly, $XT = 0$.

Now, L, T and X are sections in the ruled surface $S \xrightarrow{p} C$. Let H be a minimal section of S, with $H^2 := n$, and denote by F the class of the fibers of p, modulo the numerical equivalence $\sim$ of divisors on S. For some non-negative integers l, t, x, we have $L \sim H + lF$, $T \sim H + tF$ and $X \sim H + xF$, with $n + 2l = e$, $n + l + t = 2e$ and $n + x + t = 0$. We get $n = e - 2l$, $t = l + e$ and $x = l - 2e$. Since $t > 0$, we have $T \neq H$, hence $-x = n + t = TH \geq 0$. Therefore, $x = 0$, $l = 2e$ and $n = -3e$. Since $l > 0$, we have $L \neq H$, hence $LH \geq 0$. But $LH = -e < 0$, a contradiction.

Vice versa, assume that Q is rational of degree e, with normalization $\mathbb{P}^1 \xrightarrow{\nu} Q$. Fix a rational map $\mathbb{P}^2 \overset{\phi}{\rightsquigarrow} \mathbb{P}^1$ of algebraic degree $d \geq 2$, that satisfies $I(\phi) \cap Q = \emptyset$. The rational self-map $f := \nu\phi$ of $\mathbb{P}^2$ is degenerate with image Q, has $d(f) = de \geq 2e$, and satisfies $I(f) \cap Q = \emptyset$. Adding to the components of f generic multiples of h^2, where h is the homogeneous equation of Q, we obtain regular self-maps of $\mathbb{P}^2$ for which Q is invariant and critical. □

5. Julia set.

A domain D in a complex manifold M is *hyperbolically embedded* in M iff for any two sequences $(x_n)_n$, $(y_n)_n$ of points in D, if $x_n \underset{n}{\longrightarrow} x \in M$, $y_n \underset{n}{\longrightarrow} y \in M$, and $d_D(x_n, y_n) \underset{n}{\longrightarrow} 0$, then $x = y$. Here, d_D denotes the Kobayashi pseudo-distance.

Given two complex spaces X and Y, let $\mathrm{Hol}(X, Y)$ denote the space of holomorphic maps from X to Y, endowed with the compact-open topology. In [9] it is shown that if $D \subset M$ is relatively compact and hyperbolically embedded, then, given any complex manifold U, $\mathrm{Hol}(U, D)$ is relatively compact in $\mathrm{Hol}(U, M)$.

An irreducible complex space X is *Brody hyperbolic* iff it admits no non-constant holomorphic maps $\mathbb{C} \to X$. We will use Green's results on the hyperbolicity of the complement of a hypersurface.

Theorem 5.1 ([7]). *Let M be a compact complex manifold and $H \subset M$ a hypersurface, with irreducible components H_i, $1 \leq i \leq m$. Assume that:*

1. *$M \setminus H$ is Brody hyperbolic;*
2. *$\bigcap_{i \in I} H_i \setminus \bigcup_{j \in J} H_j$ is Brody hyperbolic for every partition $I \sqcup J = \{1, \cdots, m\}$.*

Then $M \setminus H$ is completely hyperbolic, and hyperbolically embedded in M.

Theorem 5.2 ([6], [10]). *Let M be a projective manifold and $\mathbb{C}^r \xrightarrow{f} M$ a holomorphic map that omits at least $\dim(M)+2$ ample hypersurfaces in M. Then the image of f is contained in some hypersurface of M.*

A quasi-projective irreducible curve Q_0 is *hyperbolic* iff its normalization $\tilde{Q}_0$ is a hyperbolic Riemann surface. Q_0 is hyperbolic iff there are no nonconstant maps from $\mathbb{C}$ to Q_0. We will use the following result.

Theorem 5.3 ([1]). *Let $\mathcal{C}$ be a plane curve with at least four irreducible components, at least one of them irrational. Then there are at most finitely many irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus \mathcal{C}$ is not hyperbolic.*

Given a self-map $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^n$, $J(f)$ denotes its Julia set. When $n = 1$, all but at most two points have the property that $J(f)$ is contained in the closure of their backward orbit. When $n = 2$, it may happen that no point has this property. (For example, when f has a chaotic completely invariant line.)

Theorem 5.4. *If an elliptic plane curve Q is invariant for a map $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ with $d(f) > 1$, then $J(f)$ equals the closure of the backward f-orbit of any point on Q.*

Proof. For $n \geq 0$, let $Q_n := f^{-n}(Q)$. Denote by $\mathcal{J}$ the closure of $\bigcup_{n\geq 0} Q_n$. Since Q is elliptic, $\mathcal{J}$ is the closure of the backward orbit of any point on Q, and $\mathcal{J} \subset J(f)$. We need to show that the sequence of iterates of f is normal on $\Omega = \mathbb{P}^2 \setminus \mathcal{J}$. It suffices to find some n_0 such that $\mathbb{P}^2 \setminus Q_{n_0}$ is hyperbolically embedded in $\mathbb{P}^2$.

The irreducible components of Q_n are mapped by f^n to Q, hence they are irrational. Note that Q_n has at least $n+1$ irreducible components. This can easily be seen by induction, since no irrational plane curve is completely f-invariant ([5]).

By Theorem 5.3, we deduce that when $n \geq 3$ there are at most finitely many irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus Q_n$ is not hyperbolic.

We show that for every irreducible curve $B \subset \mathbb{P}^2$ there exists some positive integer n so that $B \setminus Q_n$ is hyperbolic. Assume this is not true, and let $\mathcal{B}$ denote the finite set of irreducible curves $B \subset \mathbb{P}^2$ with the property that $B \setminus Q_n$ is not hyperbolic for all $n \geq 0$. Since $f(B \setminus Q_{n+1}) \subset f(B) \setminus Q_n$, f acts on $\mathcal{B}$. Pick an f-periodic curve $B \in \mathcal{B}$. Replacing f by an iterate, we may assume that $f(B) = B$. Let $B \xrightarrow{g} B$ denote the map induced by f. Note that B is rational, since any Zariski open subset of an irrational curve is hyperbolic. Let $\mathbb{P}^1 \xrightarrow{\nu} B$ be a normalization, and $\mathbb{P}^1 \xrightarrow{\tilde{g}} \mathbb{P}^1$ the lifting of g through ν. The backward $\tilde{g}$-orbit of $\nu^{-1}(B \cap Q)$ contains at most two points, $\mathbb{P}^1 \setminus \{\text{three points}\}$ being hyperbolic. Replacing f by an iterate, we may assume that the points of $\nu^{-1}(B \cap Q)$ are completely $\tilde{g}$-invariant, hence critical for $\tilde{g}$.

Now, for all $p \in B \cap Q$, $T_p(B) \neq T_p(Q)$. Indeed, let $(\tilde{B}, p)$ be a local irreducible component of (B,p), and $(\tilde{Q}, p)$ one of (Q,p). Since $g(\tilde{B}, p) = (\tilde{B}, p)$ and $d\tilde{g}(\nu^{-1}(p)) = 0$, Lemma 3.1 (2) implies that $df(p)_{|T_p(B_1)} = 0$. Since $f(Q,p) \subset (Q,p)$ and Q is elliptic, Lemma 3.3 implies that $df(p)_{|T_p(\tilde{Q})} \neq 0$. Consequently, $T_p(\tilde{Q}) \neq T_p(\tilde{B})$.

The local intersection number of B and Q at any $p \in B \cap Q$ is then $m_p(B,Q) = m_p(B)m_p(Q)$. By Bézout, $\deg(B)\deg(Q) = \sum_{p \in B\cap Q} m_p(B)m_p(Q)$. For all $p \in B$, $m_p(B) < \deg(B)$. Given two distinct points p and q in B, $m_p(B) + m_q(B) \leq \deg(B)$.

Recall that $B \cap Q$ consists of either one or two points. In both cases we immediately get a contradiction.

This means that $\mathcal{B}$ is empty. It follows that, for some large enough n_0, $B \setminus Q_{n_0}$ is hyperbolic for all irreducible curves $B \subset \mathbb{P}^2$.

By Theorems 5.2 and 5.1, $\mathbb{P}^2 \setminus Q_{n_0}$ is hyperbolically embedded in $\mathbb{P}^2$. □

We will use the following simple remark, to generate computer pictures of Julia sets as basin boundaries.

Remark 5.5. Assume that $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ leaves invariant an elliptic plane curve Q and a line L. If f has an attracting point $a \in L$ and if some point $r \in Q \cap L$ is repelling for the restriction of f to L, then $J(f)$ equals the boundary of the basin of a.

Proof. Denote by $L \xrightarrow{\phi} L$ the restriction of f, and by $\mathcal{A}$ the basin of a. It is clear that $J(f) \supset \partial\mathcal{A}$. Since $r \in J(\phi)$, we deduce that $r \in \partial\mathcal{A}$. The backward f-orbit of r is dense in $J(f)$, hence $J(f) \subset \partial\mathcal{A}$. □

6. Smooth cubics.

6.1. **Invariance.** Given a smooth cubic C, pick the group structure $\mathbb{C} \xrightarrow{[\cdot]} (C, +, 0)$ so that $s(\mathcal{O}_C(1)) = 0$. Three points on C are collinear when their sum is 0, and 0 is a flex of C.

Proposition 6.1. *Given a smooth plane cubic C and a multiplier $0 \neq m \in \mathbb{Z}(C)$, $R_C(m) = \{[n] \in C : 3[\overline{m}n] = c_m\}$, hence $r_C(m) = 9|m|^2$. Given $[n] \in R_C(m)$, the self-map $C \xrightarrow{g} C$, $g[t] = [mt - n]$, admits regular extensions to $\mathbb{P}^2$. Moreover, the flexes of C are pre-periodic for g.*

Proof. The first statement follows from Remark 2.4 and Remark 3.4. Since $C_3 \cup g(C_{3|m|^2}) \subset C_{3|m|^2}$, the flexes of C are pre-periodic for g. When $|m|^2 \geq 3$, the generic extension of g is regular. Lemma 6.2 below concludes the proof when $|m|^2 = 2$. □

6.2. **Algebraic degree** 2. We find in this subsection the self-maps of $\mathbb{P}^2$ of algebraic degree 2 that leave invariant a smooth cubic.

Lemma 6.2. *If a rational self-map f of $\mathbb{P}^2$ with $d(f) = 2$ leaves invariant a smooth plane cubic, then f is regular.*

Proof. Let C be an f-invariant smooth cubic, and $C \xrightarrow{g} C$ the restriction of f.

We see that f does not contract curves. Indeed, assume that E is an irreducible plane curve so that $f(E \setminus I(f))$ is a point $q \in \mathbb{P}^2$. Since $E \cap C \neq \emptyset$, $q \in C$. Since E is contracted by f, $\deg(E) \leq d(f)$, hence E is a line or a conic. Given $p \in E \cap C$, $df(p)_{|T_pE} = 0$ and $df(p)_{|T_pC} \neq 0$. Therefore, E meets C transversely. Since $E \cap C \subset g^{-1}(q)$, we get $2 = \deg(g) \geq \deg(E)\deg(C) \geq 3$, a contradiction.

Clearly, $f(\mathbb{P}^2 \setminus I(f))$ is Zariski dense in $\mathbb{P}^2$. (Otherwise, $f(\mathbb{P}^2 \setminus I(f)) = C$. Given a line L in $\mathbb{P}^2$, f would induce an isomorphism from L onto C.)

Let Q be an irreducible component of $f^*C - C$. Then $\deg(Q) \leq 3$. Since f induces a surjective map from Q to C, Q is irrational. Therefore, Q is a smooth cubic, and $f^*C = C + Q$. The support of f^*C must contain $I(f)$, hence $Q \neq C$ and $I(f) \subset Q$. Let $C \xrightarrow{i} \mathbb{P}^2$ and $Q \xrightarrow{j} \mathbb{P}^2$ be the inclusion maps, and $Q \xrightarrow{h} C$ the restriction of f. Given a line L in $\mathbb{P}^2$, the divisors j^*f^*L and h^*i^*L coincide on

$Q \setminus I(f)$. When $L \cap h(I(f)) = \emptyset$, the support of h^*i^*L does not meet $I(f)$, hence $j^*f^*L = h^*i^*L + \sum_{i\in I(f)} n_i(L)(i)$, with $0 < n_i(L) \in \mathbb{Z}$.

Assume $I(f) \neq \emptyset$. Taking degrees, we get $\deg(h) = 1$ and $\sum_{i\in I(f)} n_i(L) = 3$. Now, fix a point $q \in C \setminus (f(C \cap Q) \cup h(I(f)))$, and two lines L and M in $\mathbb{P}^2$ that pass through q and do not meet $h(I(f))$. Then $(f^*L)\cdot(f^*M) \geq g^*(q) + h^*(q) + \sum_{i\in I(f)} (i)$, hence $I(f)$ consists of one point, $I(f) = \{i\}$, and f^*L and f^*M meet transversely at i. Since $n_i(L) = 3 = n_i(M)$, we get $T_i(f^*L) = T_i(Q) = T_i(f^*M)$, hence f^*L and f^*M are tangent at i. This contradiction shows that f is regular. □

Given $C \subset \mathbb{P}^2$, let $\mathrm{Aut}_C(\mathbb{P}^2) := \{A \in \mathrm{Aut}(\mathbb{P}^2) : \ A(C) = C\}$.

Lemma 6.3. *Let $\mathcal{C}$ be the set of pairs (f, C), where f is a regular self-map of $\mathbb{P}^2$ with $d(f) = 2$, and C is an f-invariant smooth plane cubic. Let* $\mathrm{Aut}(\mathbb{P}^2)$ *act by conjugation on $\mathcal{C}$, $A \star (f, C) = (AfA^{-1}, AC)$. Then* $\mathrm{card}\big(\mathcal{C}/\mathrm{Aut}(\mathbb{P}^2)\big) = 20$.

Proof. Given λ in the Siegel figure, pick a cubic $C_\lambda \simeq \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\lambda)$. Fix $(f, C) \in \mathcal{C}$. There exist λ and $A \in \mathrm{Aut}(\mathbb{P}^2)$ with $A(C) = C_\lambda$, hence we may assume $C = C_\lambda$. Let $\mathcal{F}$ be the set of flexes of C. Pick a group structure $\mathbb{C} \xrightarrow{[\cdot]} (C, +, 0)$ with $[0] \in \mathcal{F}$. Let $m := m_C(f)$ and $[n] := -f[0]$. Since $|m|^2 = 2$, $\lambda \in \{\mathrm{i}, \mathrm{i}\sqrt{2}, \frac{1+\mathrm{i}\sqrt{7}}{2}\}$, and $m \in \mathcal{M}(\lambda)$, where $\mathcal{M}(\mathrm{i}) = \{\pm 1 \pm \mathrm{i}\}$, $\mathcal{M}(\mathrm{i}\sqrt{2}) = \{\pm\mathrm{i}\sqrt{2}\}$, and $\mathcal{M}(\frac{1+\mathrm{i}\sqrt{7}}{2}) = \{\frac{\pm 1\pm\mathrm{i}\sqrt{7}}{2}\}$. By Remark 3.4, $[n] \in \mathcal{N}(m) := \big([\frac{1}{2}] + \mathcal{F}\big) \cup \big([\frac{1-m}{2}] + \mathcal{F}\big)$. Identify with $\mathcal{N}(m)$ the set of self-maps of $\mathbb{P}^2$ of algebraic degree 2 that leave C invariant and have multiplier m on C. Similarly, $\mathrm{Aut}_C(\mathbb{P}^2)$ is identified with $\mathcal{A} := \mathbb{U}(C) \times \mathcal{F}$, $A = (m_C(A), -A[0])$. Now, $\mathcal{A}$ acts by conjugation on $\mathcal{N}(m)$, $(u, [v]) \star [n] = [un + (1-m)v]$. When $\lambda = \mathrm{i}$, $\mathcal{A} \star [\frac{1}{2}] = \mathcal{N}(m)$. When $\lambda = \frac{1+\mathrm{i}\sqrt{7}}{2}$, the orbits of $\mathcal{A}$ on $\mathcal{N}(m)$ are $\mathcal{A} \star [\frac{1}{2}]$, $\mathcal{A} \star [\frac{1-m}{2}]$. When $\lambda = \mathrm{i}\sqrt{2}$, the orbits of $\mathcal{A}$ on $\mathcal{N}(m)$ are $\mathcal{A}\star[\frac{1}{2}]$, $\mathcal{A}\star[\frac{1-m}{2}]$, $\mathcal{A}\star[\frac{1}{6}]$, $\mathcal{A}\star[\frac{1-3m}{6}]$. □

Given $\lambda \in \{\mathrm{i}, \mathrm{i}\sqrt{2}, \frac{1+\mathrm{i}\sqrt{7}}{2}\}$, $m \in \mathcal{M}(\lambda)$, $[n] \in \mathcal{N}(m)$ as in the proof of Lemma 6.3, let $f_{m,[n]}$ be the extension to $\mathbb{P}^2$ of the self-map $[z] \mapsto [mz - n]$ of C_λ.

Given two points $p \neq q$ in $\mathbb{P}^2$, let $L(p, q)$ be the line passing through p and q. As usual, $\check{\mathbb{P}}^2 \simeq \mathbb{P}^2$ denotes the space of lines in $\mathbb{P}^2$.

Lemma 6.4. *Given a self-map f of $\mathbb{P}^2$ with $d(f) = 2$ that leaves invariant a smooth cubic $C \subset \mathbb{P}^2$, let $\mathcal{L} := \{L \in \check{\mathbb{P}}^2 : \ \deg(f(L)) = 1\}$, and $\mathcal{P} := \{f(L) : \ L \in \mathcal{L}\}$. Then $\mathcal{L}$ and $\mathcal{P}$ are smooth cubics in $\check{\mathbb{P}}^2$, and $\mathcal{L}$ is isomorphic to C. Moreover, $\mathcal{L}$ and $\mathcal{P}$ have the same set $\mathbb{F}$ of flexes, $\mathbb{F}$ depends only on C, and $\{f(L) : \ L \in \mathbb{F}\} = \mathbb{F}$. Finally, $\mathcal{L} = \mathcal{P}$ iff $\mathcal{P}$ is isomorphic to C iff f has either 2 or 4 invariant lines.*

Proof. We may assume that $C = C_\lambda$, with $\lambda \in \{\mathrm{i}, \mathrm{i}\sqrt{2}, \frac{1+\mathrm{i}\sqrt{7}}{2}\}$, and $f = f_{m,[n]}$, with $m \in \mathcal{M}(\lambda)$ and $[n] \in \mathcal{N}(m)$. We see that $\mathcal{L} = \{L(p, q) : \ p \neq q \ \& \ f(p) = f(q)\}$, hence $\mathcal{L}$ is a curve of degree 3 in $\check{\mathbb{P}}^2$. Let $L[z] := L([z + \frac{1}{2}], [z + \frac{1+\overline{m}}{2}]) = L[z + \frac{\overline{m}}{2}]$. One of the components of $\mathcal{L}$ is the elliptic curve $\mathcal{L}_0 := \{L[z] : \ [z] \in C\}$, hence $\mathcal{L} = \mathcal{L}_0$ is a smooth cubic in $\check{\mathbb{P}}^2$. The map $\mathcal{L} \xrightarrow{\psi} C$, $\psi(L[z]) = [mz]$, is an isomorphism. Since $L[\frac{1}{2}] \cap L[\frac{\overline{m}}{4}] \cap L[\frac{1}{2} + \frac{\overline{m}}{4}] = \{[0]\}$, $\sigma(\psi_*\mathcal{O}_{\mathcal{L}}(1)) = [0]$. Therefore, three lines $L[z_j]$ concur iff $[\sum_j z_j] \in \{[0], [\frac{\overline{m}}{2}]\}$. It follows that the set of flexes of $\mathcal{L}$ is $\mathbb{F} := \{L[z] : \ [z] \in \mathcal{F}\}$. When $[z] \in \mathcal{F}$, $L[z] = L([z + \frac{1}{2}], [z + \frac{\lambda}{2}])$, hence $\mathbb{F}$ does not depend on m or $[n]$.

Let $P[z] := L([z + \frac{m}{2} - n], [z + \frac{m}{2} + 2n])$. Then $f(L[z]) = P[mz] = f(L[z + \frac{3\overline{m}n}{2}])$, and $\mathcal{P} = \{P[z] :\ [z] \in C\}$, a smooth cubic in $\check{\mathbb{P}}^2$. Let $C \xrightarrow{\widehat{\ }} \widehat{C}$ be the quotient map associated to the translation $[z] \mapsto [z + 3n]$. The map $\mathcal{P} \xrightarrow{\widetilde{\psi}} \widehat{C}$, $\widetilde{\psi}(P[z]) = \widehat{[z]}$, is an isomorphism. Since $P[n - \frac{m}{2}] \cap P[-\frac{n}{2}] \cap P[\frac{m-n}{2}] = \{[0]\}$, $\sigma(\widetilde{\psi}_* \mathcal{O}_{\mathcal{P}}(1)) = \widehat{[0]}$. It follows that three lines $P[z_j]$ concur iff $[\sum_j z_j] \in \{[0], [3n]\}$, and then the set of flexes of $\mathcal{P}$ is also $\mathbb{F}$. There are five cases to consider:

1. $\lambda = \mathrm{i}$ and $[n] = [\frac{1}{2}]$,
2. $\lambda = \mathrm{i}\sqrt{2}$ and $[n] \in \{[\frac{1}{2}], [\frac{1-m}{2}]\}$,
3. $\lambda = \mathrm{i}\sqrt{2}$ and $[n] \in \{[\frac{1}{6}], [\frac{1-3m}{6}]\}$,
4. $\lambda = \frac{1+\mathrm{i}\sqrt{7}}{2}$ and $[n] = [\frac{1}{2}]$,
5. $\lambda = \frac{1+\mathrm{i}\sqrt{7}}{2}$ and $[n] = [\frac{1-m}{2}]$.

For all $[z] \in \mathcal{F}$, $P[z] = L[z + \frac{1}{3}]$ in case (3), and $P[z] = L[z]$ in all other cases. It follows that f permutes the lines in $\mathbb{F}$. For further reference, let $\mathbb{F} \xrightarrow{F} \mathbb{F}$ denote this permutation, and define $\mathcal{F} \xrightarrow{\phi} \mathcal{F}$, $\phi := \psi F \psi^{-1}$. Then $\phi[z] = [mz + \tau]$, where $[\tau] = [\frac{m}{3}]$ in case (3), and $[\tau] = [0]$ in the other four cases. Note that $\mathcal{L} = \mathcal{P}$ iff $[\frac{\overline{m}}{2}] = [3n]$ iff $\lambda = \frac{1+\mathrm{i}\sqrt{7}}{2}$ and $[n] = [\frac{1-m}{2}]$, which is case (5). In this case, f has $|m-1|^2$ invariant lines. In the other four cases, let $\underline{l}$ be the sequence of lengthes of the f-cycles of lines, ordered increasingly. Then $\underline{l} = (1, 8)$ in the cases (1) and (4), $\underline{l} = (1, 1, 1, 2, 2, 2)$ in case (2), and $\underline{l} = (3, 6)$ in case (3). □

Lemma 6.5. *There are no self-maps of* $\mathbb{P}^2$ *of algebraic degree* 2 *that leave invariant two smooth cubics.*

Proof. We keep the notations from (the proof of) Lemma 6.4. Assume that f leaves invariant a smooth cubic $C' \neq C$. Since $C \simeq \mathcal{L} \simeq C'$, there exists $A_1 \in \mathrm{Aut}(\mathbb{P}^2)$ with $A_1(C') = C$. Since $A_1 f A_1^{-1}$ leaves C invariant, there is $A_2 \in \mathrm{Aut}_C(\mathbb{P}^2)$ with $A_2 A_1 f A_1^{-1} A_2^{-1} = f_{m',[n']} := f'$, where $m' \in \mathcal{M}(\lambda)$, $[n'] \in \mathcal{N}(m')$. If $A := A_2 A_1$, then $A f A^{-1} = f'$ and $A(C) \neq C$. Given $B \in \mathrm{Aut}_C(\mathbb{P}^2)$ with $Bf = fB$, we may replace A by AB. In the cases (1), (2), (4) and (5), we have $fM = Mf$, where $M \in \mathrm{Aut}_C(\mathbb{P}^2)$, $M[z] := [-z]$. In the cases (2) and (3), we have $fT = Tf$, where $T \in \mathrm{Aut}_C(\mathbb{P}^2)$, $T[z] := [z - \frac{m+1}{3}]$.

Let $\mathcal{L}'$ be the curve of lines that are mapped to lines by f', and similarly define $\mathcal{P}'$, F', ψ' and ϕ'. Clearly, $\{A(L) :\ L \in \mathcal{L}\} = \mathcal{L}'$, and $\{A(P) :\ P \in \mathcal{P}\} = \mathcal{P}'$. Define $\mathcal{L} \xrightarrow{\alpha} \mathcal{L}'$, $\alpha(L) := A(L)$. Then $\alpha(\mathbb{F}) = \mathbb{F}$, and $\alpha F = F' \alpha$ on $\mathbb{F}$. Define $C \xrightarrow{\beta} C$, $\beta := \psi' \alpha \psi^{-1}$. Then $\beta(\mathcal{F}) = \mathcal{F}$, and $\beta\phi = \phi'\beta$. Write $\beta[z] = [uz - v]$, with $u \in \mathbb{U}(C)$ and $[v] \in \mathcal{F}$. Then $[u(mz + \tau) - v] = [m'(uz - v) + \tau']$ for all $[z] \in \mathcal{F}$. We get $m = m'$, hence $\mathcal{L} = \mathcal{L}'$, $\psi = \psi'$, and $[(m-1)v] = [\tau' - u\tau]$. It suffices to show $\beta = 1_C$, since then $\alpha = 1_{\mathcal{L}}$, and $A = 1_{\mathbb{P}^2}$, contradicting $A(C) \neq C$.

In the cases (5) and (4), $[\tau] = [0] = [\tau']$, hence $[v] = 0$. Since $fM = Mf$, we may assume $u = 1$, i.e. $\beta = 1_C$.

In the cases (3) and (2), $[\tau' - u\tau] \in \mathbb{Z}[\frac{m}{3}]$ and $[(m-1)v] \in \mathbb{Z}[\frac{m-1}{3}]$, hence $[\tau' - u\tau] = [0]$ and $[v] \in \mathbb{Z}[\frac{m+1}{3}]$. Since $fT = Tf$, we may assume that $[v] = 0$, and then $[\tau'] = [u\tau]$. In case (3), we get $[\tau'] = [\frac{m}{3}] = [\tau]$ and $u = 1$, hence $\beta = 1_C$. In case (2), since $fM = Mf$, we may assume $u = 1$, i.e. $\beta = 1_C$.

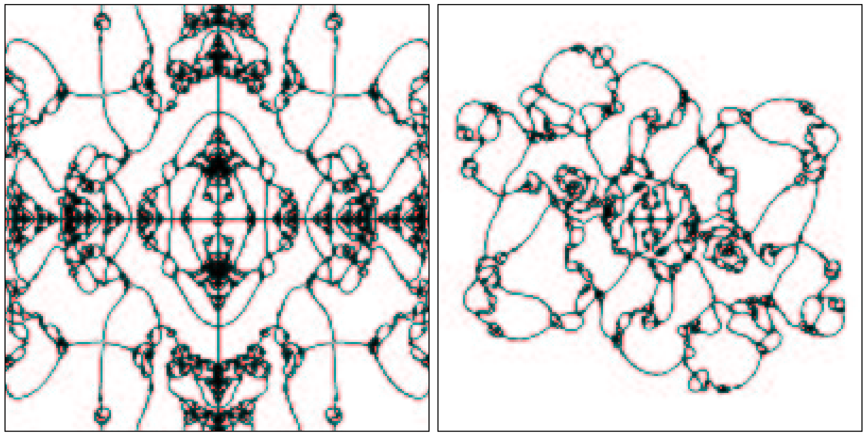

FIGURE 1. Self-map of degree 2 with an invariant smooth cubic.

In case (1), $f = f'$, $[\tau] = 0$, $[v] = 0$. Since $L[0]$ is the only f-invariant line, $\beta[0] = [0]$. Since $L[\frac{\overline{m}}{4}] = f^*L[0] - L[0]$, $\beta[\frac{1}{2}] = [\frac{1}{2}]$. Therefore, $u = \pm 1$. Since $fM = Mf$, we may assume $u = 1$, i.e. $\beta = 1_C$. □

Proposition 6.6. *Up to conjugation by a Möbius transformation, there are* 20 *self-maps of* $\mathbb{P}^2$ *of algebraic degree* 2 *with an invariant smooth cubic.*

Proof. Assume $Af_{m,[n]}A^{-1} = f_{m',[n']}$. By Lemma 6.5, $AC_\lambda = C_{\lambda'}$, hence $\lambda = \lambda'$. It follows that $m = m'$, and then $[n] = [n']$. □

Remark 6.7. Given $(f, C) \in \mathcal{C}$, the ramification divisor R of f is a smooth cubic isomorphic to $\mathcal{P}$, and $R \cap C = \mathcal{F}$. The branching curve $B := f(R)$ is the dual of $\mathcal{P}$.

Proof. Note that R is a cubic in $\mathbb{P}^2$. Since $f(L[z]) = f(L[z + \frac{3\overline{m}n}{2}])$, the map $C \xrightarrow{r} R$, $r[z] := L[z] \cap L[z + \frac{3\overline{m}n}{2}]$, is well-defined. Its fibers are the orbits of the translation $\tau[z] = [z + \frac{3\overline{m}n}{2}]$, and the same is true for the map $C \ni [z] \mapsto P[mz] \in \mathcal{P}$. Therefore, $R \simeq C/\tau \simeq \mathcal{P}$. When $[z] \in \mathcal{F}$, $r[z + \frac{1}{2}] = [z]$, hence $R \cap C = \mathcal{F}$.

Given $x \in R$, the differential $df(x)$ has 1-dimensional kernel L_x, or else $\mathcal{L}$ would contain the pencil of lines through x. Clearly, $L_x \in \mathcal{L}$. Let $L \neq L_x$ be another line in $\mathcal{L}$ that passes through x. If x is not critical for the restriction of f to R, then $f(L) = T_{f(x)}B$. Therefore, $\mathcal{P}$ is the dual of B. □

Remark 6.8. When $\lambda = \frac{1+i\sqrt{7}}{2}$ and $[n] = [\frac{1-m}{2}]$, we have $\mathcal{L} = \mathcal{P}$, i.e. $f := f_{m,[n]}$, $m = \frac{\pm 1 \pm \sqrt{7}}{2}$, leaves invariant the smooth cubic of lines $\mathcal{L}$, inducing on $\mathcal{L}$ the self-map $g(L[z]) = L[mz]$. The map $\mathcal{L} \times \mathcal{L} \xrightarrow{\pi} \mathbb{P}^2$, $\pi(L_1, L_2) = L_1 \cap L_2$, is regular, and $\pi(g, g) = f\pi$. It follows that f is strictly critically finite, with $J(f) = \mathbb{P}^2$. The ramification R is isomorphic to C, and the branching B is the dual of $\mathcal{L}$. Moreover, $f^*B = B + 2R$. Such maps with an invariant smooth cubic of lines appear in Proposition 7.2. The self-maps with an invariant curve of plane curves are discussed in [4].

Example 6.9. The smooth cubic $y^2z = x(x - z)(x - b^2z)$, where $b = 1 + \sqrt{2}$, is given in the Siegel figure by $i\sqrt{2}$. Pick the origin at $[0, 1, 0]$. Then $f_{i\sqrt{2},[1,0,1]} =$

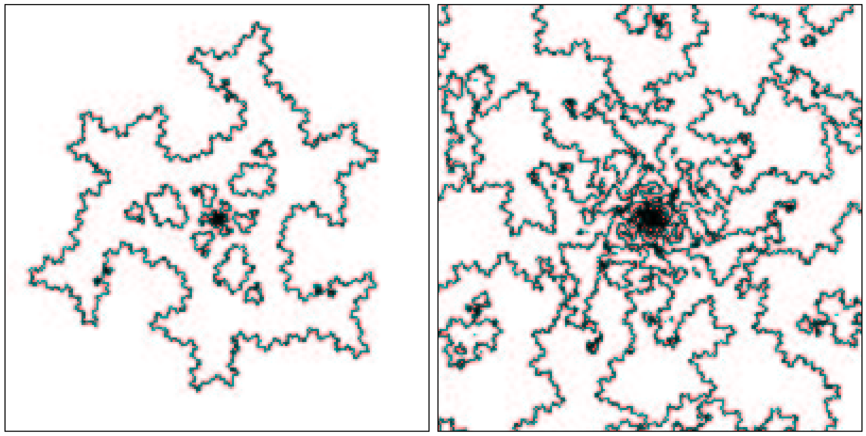

FIGURE 2. Tangent process on the Fermat cubic.

$[y^2 + (3b+2)x^2 - 2b^2xz - b^3z^2, -2(b+1)\mathrm{i}y(x+bz), y^2 - (b-2)x^2 + 2b^2xz - b^3z^2]$. This self-map has five basins of attraction; by Remark 5.5, its Julia set equals their common boundary. Figure 1 shows the traces of the Julia set on the lines $(x = 0)$ (left) and $(x = 0.01\mathrm{i}z)$ (right), near the flex $[0, 1, 0]$.

6.3. **Tangent processes.** The classic "tangent process" on a smooth plane cubic C maps $p \in C$ to the residual intersection $g(p)$ of C with the tangent line $T_p(C)$. Clearly, $m(g) = -2$, and g admits regular extensions to $\mathbb{P}^2$.

Definition 6.10. A *tangent process* on a (possibly singular) elliptic plane curve Q is a self-map of Q with multiplier -2 that admits regular extensions to $\mathbb{P}^2$.

In suitable coordinates, the classic tangent processes are given by Desboves' tangent formula. Let $[x_j]$ be the coordinates in $\mathbb{P}^2$, $j \in \mathbb{Z}/3$. When $k^3 \neq 1$, the cubic $C_k := (h_k = 0)$, $h_k := \sum_j x_j^3 - 3k\prod_j x_j$, is smooth. Every elliptic curve is isomorphic to some C_k. The map $D[x_j] := [x_j(x_{j+1}^3 - x_{j+2}^3)]$ is a rational extension of the classic tangent process g_k on C_k. Self-maps that leave invariant arbitrarily many of the curves C_k can be obtained by adding multiples of $\prod_k h_k$ to the components of a large iterate of D. The extensions to $\mathbb{P}^2$ of g_k are parametrized by 3×3 matrices $A = (a_{jl})$ of complex numbers, $D_{k,A}[x_j] = [x_j(x_{j+1}^3 - x_{j+2}^3) + (\sum_l a_{jl}x_j)\, h_k]$. By Remark 2.5, $I(D)$ can be used to get regular extensions of g_k with attracting points.

Example 6.11. When the matrix $A := \mathrm{diag}(a_j)$ is diagonal, the map $D_{k,A}$ commutes with $[x_j] \mapsto [\exp\left(\frac{2\pi \mathrm{i} j}{3}\right) x_j]$, and the lines $(x_j = 0)$ are $D_{k,A}$-invariant. If $\max(|a_{j+1} - 1|, |a_{j+2} + 1|) < |a_j|$, the fixed point $P_j := (x_{j+1} = 0 = x_{j+2})$ is attracting. If, further assumed, $\max(|2 + 3(a_j - a_{j+1})|, |2 + 3(a_{j+2} - a_j)|) > 1$, Remark 5.5 implies that $J(D_{k,A})$ is the boundary of the basin of P_j. For example, when $A := \mathrm{diag}\,(\mathrm{i}, 2\mathrm{i} - 1, 1 - 2\mathrm{i})$, the Julia set $J(D_{k,A})$ is the common boundary of the basins of $[0, 1, 0]$ and $[0, 0, 1]$. Figure 2 shows the trace of $J(D_{0,A})$ on the line $(x + y = 0)$ near $[1, -1, 0]$ (left), and a zoom-in at the center (right).

When $A = \mathrm{diag}(a, 1, -1)$, $a \neq 0$, $D_{0,A}$ leaves invariant the Fermat cubic C_0 and the pencil of lines through $[1, 0, 0]$. We discuss such maps in the next subsection.

6.4. **Elementary maps with an invariant smooth cubic.** Given $P \in \mathbb{P}^2$, a map $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ with $d(f) > 1$ is *elementary* with *center* P iff it leaves invariant the pencil $\check{P}$ of lines passing through P, i.e. $f(L) \in \check{P}$ for all $L \in \check{P}$.

Lemma 6.12. *Let C be a smooth elliptic curve, and $C \xrightarrow{q} \mathbb{P}^1$ a non-constant elliptic function. Assume that a self-map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$ with $\deg(g) > 1$ lifts through q to a self-map $C \xrightarrow{h} C$, $qh = gq$. Then there exists a smooth elliptic curve D, a map $C \xrightarrow{r} D$, and a finite nontrivial group $G < \mathrm{Aut}(D)$, with associated quotient map $D \xrightarrow{s} D/G = \mathbb{P}^1$, so that $q = sr$.*

Proof. Let $B \subset \mathbb{P}^1$ be the branch locus of q, and $R = q^{-1}(B)$. Then $C \backslash R \xrightarrow{q} \mathbb{P}^1 \backslash B$ is an unramified covering, of degree $n+1$. Let $m := m(h)$. Since $|m| > 1$, we can find an h-periodic point $c_0 \in C \setminus R$. Let c_i, $1 \le i \le n$, be the other points in the fiber $q^{-1}(q(c_0))$. Replacing g by an iterate, we may assume that $h(c_0) = c_0$ and $h^2(c_i) = h(c_i)$. Let $(C, c_0) \xrightarrow{\gamma_i} (C, c_i)$ be the transition maps on the sheets of q, i.e. $q\gamma_i = q$. Since $h^2\gamma_i(c_0) = h\gamma_i h(c_0)$ and $q(h^2\gamma_i) = q(h\gamma_i h)$, we get $h^2\gamma_i = h\gamma_i h$.

Fix a universal covering $\mathbb{C} \xrightarrow{u} C$ with $u(0) = c_0$. Then h lifts through u to the linear map $\mathbb{C} \xrightarrow{H} \mathbb{C}$, $H(z) = mz$. Choose $z_i \in u^{-1}(c_i)$, and let $(\mathbb{C}, 0) \xrightarrow{\Gamma_i} (\mathbb{C}, z_i)$ be the lifting of γ_i through u. Let $\Gamma_i(z) = z_i + m_i z + \sum_{j \ge 2} m_{i,j} z^j$ be the Taylor series at 0. Since $|m| > 1$ and $uH^2\Gamma_i = uH\Gamma_i H$ near 0, we get $m_{ij} = 0$ for all $j \ge 2$. Therefore, $\mathbb{C} \xrightarrow{\Gamma_i} \mathbb{C}$ are affine, $\Gamma_i(z) = m_i z + z_i$. Let $\mathbb{C} \xrightarrow{\Gamma_0} \mathbb{C}$ be the identity map.

Identify the lattice $\Omega = u^{-1}(c_0)$ with a group of translations on $\mathbb{C}$, C with $\mathbb{C}/\Omega$, and $\mathbb{C} \xrightarrow{u} C$ with the quotient map associated to the action of Ω on $\mathbb{C}$. Consider the map $\mathbb{C} \xrightarrow{Q} \mathbb{P}^1$, $Q = qu$, and the set $\mathcal{A} = \{\mathbb{C} \xrightarrow{\Gamma} \mathbb{C} : \ Q\Gamma = Q\}$. Clearly, $\mathcal{A} = \bigcup_{0 \le i \le n} \Omega\Gamma_i$ and this union is disjoint. Therefore, $\mathcal{A}$ is a group of affine self-maps of $\mathbb{C}$, and Ω is a subgroup of $\mathcal{A}$ of index $[\mathcal{A} : \Omega] = n+1$. Note that $\mathcal{A}(0) = Q^{-1}(Q(0))$, and that the evaluation map $\mathcal{A} \ni \Gamma \mapsto \Gamma(0) \in \mathcal{A}(0)$ is bijective.

The translations in $\mathcal{A}$ form a normal subgroup $\mathcal{T}$ of $\mathcal{A}$, with $\Omega \le \mathcal{T}$. Since Q is non-constant, $\mathcal{T} \subset \mathcal{A}(0)$ is discrete in $\mathbb{C}$, hence $\mathcal{T}$ is a lattice in $\mathbb{C}$. Consider the elliptic curve $D = \mathbb{C}/\mathcal{T}$, with quotient map $\mathbb{C} \xrightarrow{v} D$. Let $C \xrightarrow{s} D$ be the map induced by v through u, $v = ru$. The group $G = \mathcal{A}/\mathcal{T}$ is a finite group of automorphisms of D, of order $k \le n+1$. Let $D \xrightarrow{s} D/G$ be the quotient map.

Clearly, Q induces through v a map $D \xrightarrow{\alpha} \mathbb{P}^1$, $Q = \alpha v$. We see that $\deg(\alpha) = k$. (Indeed, $\alpha^{-1}(Q(0))$ has cardinality k, and $Q(0)$ is not a branch point of α.) Clearly, α induces through s a map $D/G \xrightarrow{\beta} \mathbb{P}^1$, $\beta s = \alpha$. Since $\deg(\alpha) = k = \deg(s)$, β is an isomorphism. We identify, via β, the map α with the quotient map s. We get $qu = Q = \alpha v = sv = sru$, hence $q = sr$. ◻

If, in Lemma 6.12, $\deg(q)$ is a prime number, then r is an isomorphism; in this case, we may assume that $D = C$ and $q = s$.

The Fermat cubic $C_0 := (x^3 + y^3 + z^3 = 0)$ is the only smooth elliptic curve with an automorphism γ satisfying $\gamma^3 = 1$ and $\mathrm{Fix}(\gamma) \ne \emptyset$. Let $c_0 = [0, -1, 1]$. Then $\mathrm{Aut}_{c_0}(C_0) \simeq \mathbb{Z}/6$, generated by $\gamma_0[x, y, z] := [\tau x, z, y]$, with $\tau = \exp\left(\frac{2\pi \mathrm{i}}{3}\right)$. Let $G_0 =< \gamma_0^2 >$. Note that $\mathrm{Fix}(G_0) = C_0 \cdot (x = 0)$. For $c \in C \setminus (x = 0)$, $G_0(c)$ consists of three collinear points that determine a line passing through $P_0 := [1, 0, 0]$. In other words, the quotient map $C_0 \longrightarrow C_0/G_0$ can be identified with the map

$C_0 \xrightarrow{q_0} \check{P}_0$ that associates to $c \in C_0$ the line joining P_0 to c. In homogeneous coordinates, $C_0 \xrightarrow{q_0} \mathbb{P}^1$ is the central projection $q_0[x, y, z] = [y, z]$.

Put $\Omega_0 = \mathbb{Z} + \mathbb{Z}\tau$, and let $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega_0$ denote the quotient map. Fix an isomorphism $C_0 \simeq \mathbb{C}/\Omega_0$ with $[0] = c_0$. Then $\gamma_0[t] = [-\tau t]$, as follows from differentiating γ_0 at its fixed point c_0. Note that $\mathrm{Fix}(G_0) = [\frac{\tau-1}{3}\mathbb{Z}]$.

Proposition 2.3 yields the following criterion for a self-map of C_0 to admit an elementary extension with center P_0.

Corollary 6.13. *Given a map $C_0 \xrightarrow{h} C_0$ with $d := \deg(h) > 1$, h extends to an elementary map with center $P_0 \iff h(c_0) \in \mathrm{Fix}(G_0) \iff h$ commutes with G_0.*

Proof. Since $m(h) \in \Omega_0 = \mathbb{Z} + \mathbb{Z}\tau$, either $\frac{d}{3} \in \mathbb{Z}$ or $\frac{d-1}{3} \in \mathbb{Z}$, hence $d \geq 3$.

If h extends to an elementary map with center P_0, it induces through q_0 a map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$, $q_0 h = g q_0$. The critical points of q_0 are the three fixed points of G_0. Since h is unramified and c_0 is critical for q_0, $h(c_0)$ is critical for q_0, i.e. $h(c_0) \in \mathrm{Fix}(G_0)$.

Assume $h(c_0) \in \mathrm{Fix}(G_0)$. For all $\gamma \in G_0$, $h\gamma(c_0) = \gamma h(c_0)$. Since $m(h\gamma) = m(\gamma h)$, $h\gamma = \gamma h$. Therefore, h commutes with G_0.

If h commutes with G_0, it induces through q_0 a self-map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$, $g = [g_1, g_2]$, $g_j \in \mathbb{C}_d[y, z]$. Then $h^*\mathcal{O}_{C_0}(1) \simeq h^*q^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq q^*g^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq q^*\mathcal{O}_{\mathbb{P}^1}(d) \simeq \mathcal{O}_{C_0}(d)$. By Proposition 2.3, h extends to $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$, $f = [f_0, f_1, f_2]$, $f_j \in \mathbb{C}_d[x, y, z]$.

We have $q^*g^*(y = 0) = q^*(g_1 = 0) = C_0 \cdot (g_1 = 0)$, and $h^*q^*(y = 0) = h^*(C_0 \cdot (y = 0)) = C_0 \cdot f^*(y = 0) = C_0 \cdot (f_1 = 0)$, hence the divisors $(g_1 = 0)$ and $(f_1 = 0)$ leave the same trace on C_0. Therefore, there exists a constant $0 \neq \alpha_1 \in \mathbb{C}$ so that $f_1 - \alpha_1 g_1$ vanishes on C_0. Let $e = x^3 + y^3 + z^3$. Then there exists a polynomial $\beta_1 \in \mathbb{C}_{d-3}[x, y, z]$ so that $f_1 - \alpha_1 g_1 = e\beta_1$. Similarly, there exist a constant $0 \neq \alpha_2 \in \mathbb{C}$ and a polynomial $\beta_2 \in \mathbb{C}_{d-3}[x, y, z]$ so that $f_2 - \alpha_2 g_2 = e\beta_2$.

Consider the rational self-map $\tilde{f}$ of $\mathbb{P}^2$, $\tilde{f} = [f_0, \alpha_1 g_1, \alpha_2 g_2]$. Then $I(f) \subset \{c_0\}$, and $\tilde{f}_{|C_0} = f_{|C_0} = h$. When $f_0(c_0) \neq 0$, $\tilde{f}$ is an elementary extension of h. When $f_0(c_0) = 0$, $[f_0 + x^{d-3}e, \alpha_1 g_1, \alpha_2 g_2]$ is an elementary extension of h. ☐

Proposition 6.14. *Let $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ be an elementary map with center P, that leaves invariant a smooth cubic C. Then there exists a Möbius map $M \in \mathrm{Aut}(\mathbb{P}^2)$ with $M(P) = P_0$ and $M(C) = C_0$.*

Proof. Let $C \xrightarrow{h} C$ be the restriction of f to C. Clearly, $P \notin C$. Let $C \xrightarrow{q} \check{P}$ be the regular map that associates to a point $c \in C$ the line joining P to c. Then q has topological degree 3, and h induces through q a map $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$, $qh = gq$. By Lemma 6.12, there exists a group $\mathbb{Z}/3 \simeq G < \mathrm{Aut}(C)$, with $\mathrm{Fix}(G) \neq \emptyset$, whose orbits lie on lines through P. Fix $c \in \mathrm{Fix}(G)$, necessarily a flex of C. Pick an isomorphism $C \xrightarrow{m} C_0$ with $m(c) = c_0$. As in the proof of Proposition 2.3, m extends to a Möbius map $M \in \mathrm{Aut}(\mathbb{P}^2)$. Since $\mathbb{Z}/3 = MGM^{-1} < \mathrm{Aut}_{c_0}(C_0)$, $MGM^{-1} = G_0$. Since M maps lines to lines and G-orbits to G_0-orbits, $M(P) = P_0$. ☐

Example 6.15. If a map $C_0 \xrightarrow{h} C_0$ with $\deg(h) = 3$ (minimal) has an elementary extension $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ with center P_0, then $\pm m(h) \in \{1-\tau, \tau-\tau^2, \tau^2-1\}$. We see that h maps the set of flexes of C_0 onto $\mathrm{Fix}(G_0)$, and is constant on $\mathrm{Fix}(G_0)$. Therefore, up to Möbius conjugation, there are six one-parameter families of elementary self-maps of $\mathbb{P}^2$ of algebraic degree 3, that leave invariant a smooth cubic.

By Desboves' secant formula, $h[t] = [(\tau - 1)t]$ is given in projective coordinates by $h[x,y,z] = [(\tau^2-1)xyz, z^3 - \tau y^3, y^3 - \tau z^3]$. The elementary extensions f_a of h are obtained by adding $a(x^3+y^3+z^3)$, with $a \neq 0$, to the first component of h.

Let $S(f)$ be the support of the Green measure associated to a self-map $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$, and $R(f)$ the closure of the set of repelling periodic points of f.

Assume $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ is elementary with center P. Since $f^{-1}(P) = P$, P is superattracting for f, and the basin of attraction of P, denoted $\mathcal{A}(f)$, is connected.

Proposition 6.16. *If an elementary map $\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$ with center P leaves invariant a smooth cubic, then:*

1. $J(f) = S(f) = R(f) = \partial\mathcal{A}(f)$, *and* $\overline{\mathcal{A}(f)} = \mathbb{P}^2$.
2. $\bigcup_{r\geq 0} f^r(U) = \mathbb{P}^2 \setminus \{P\}$, *for every open set U with $U \cap R(f) \neq \emptyset$ and $P \notin U$.*

Proof. Let $C \xrightarrow{h} C$ be the restriction of f to an f-invariant smooth cubic C, P the center of f, $C \xrightarrow{q} \mathbb{P}^1$ the central projection from P, and $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$ the self-map induced by h through q. Let $m = m(h)$ and $d = |m|^2 > 1$. By Lemma 6.12, q is the quotient map $C \longrightarrow C/G$ associated to a group $\mathbb{Z}/3 \simeq G < \mathrm{Aut}(C)$.

Since g is strictly critically finite, $J(g) = \mathbb{P}^1$, and $\mathcal{A}(f)$ is the only Fatou component of f, hence $\overline{\mathcal{A}(f)} = \mathbb{P}^2$ iff $J(f) = \partial\mathcal{A}(f)$.

For every elementary self-map of $\mathbb{P}^2$, $R(f) = S(f)$ (cf. [3]). By Corollary 6.13, h commutes with G. Replacing f by an iterate, we may assume that f fixes a flex $x \in \mathrm{Fix}(G)$. The line F_x joining x to P is tangent to C, hence $f_x'(x) = m$, where $F_x \xrightarrow{f_x} F_x$ is the polynomial function induced by f. Also, since $q^*q(x) = 3x$, we get $g'(q(x)) = m^3$. It follows that x is repelling for f (with eigenvalues m and m^3), hence $x \in S(f)$. Since $S(f)$ is completely f-invariant, Theorem 5.4 implies $J(f) \subset S(f)$, hence $J(f) = S(f)$.

Clearly, $\partial\mathcal{A}(f) \subset J(f)$. Since x is repelling for f_x, we have $x \in \partial\mathcal{A}(f_x) \subset \partial\mathcal{A}(f)$. We get $J(f) \subset \partial\mathcal{A}(f)$, hence $J(f) = \partial\mathcal{A}(f)$, and the first statement is proved.

Let F_1 be the non-minimal Hirzebruch surface. Statement (2) follows from [3], once we show that $F_1 \xrightarrow{f} F_1$ has no completely invariant curves besides the negative section. Otherwise, f leaves completely invariant some affine section, so that f_x has two exceptional points. But then $|f_x'(x)| = d$, in contradiction with $f_x'(x) = m$. ☐

7. Dual of smooth cubic.

Given $p \in \mathbb{P}^2$, $\check{p} := \{L \in \check{\mathbb{P}}^2 \,:\, p \in L\}$ is a line in $\check{\mathbb{P}}^2$. Given a line l in $\check{\mathbb{P}}^2$, $\check{l} := \bigcap_{L \in l} L$ is a point in $\mathbb{P}^2$. For every $p \in \mathbb{P}^2$, $\check{\check{p}} = p$. For every line l in $\check{\mathbb{P}}^2$, $\check{\check{l}} = l$. Given a smooth cubic C in $\mathbb{P}^2$, with set $\mathcal{F}$ of flexes, choose the group structure $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega = C$ so that $[0] \in \mathcal{F}$. Given $m \in \mathbb{Z}$, let $C_m := \{c \in C \,:\, mc = 0\}$. The dual curve $\check{C}$, defined as the curve of tangents to C, is a sextic in $\check{\mathbb{P}}^2$ with ordinary cusps T_aC, $a \in \mathcal{F}$, normalized by $C \xrightarrow{\nu} \check{C}$, $\nu(c) := T_c(C)$. For all $c \in C$, $T_{\nu(c)}\check{C} = \check{c}$. In this way, the dual of $\check{C}$, defined as the curve of tangents to $\check{C}$, is identified with C. Given a line l in $\check{\mathbb{P}}^2$ not passing through the cusps of $\check{C}$, recall that $l_{\check{C}} := i^*l$, where $\check{C} \xrightarrow{i} \check{\mathbb{P}}^2$ denotes the inclusion map, and $l_C := \nu^* l_{\check{C}}$. We see that $l_C = \sum\{(c) \,:\, \check{l} \in T_c(C)\}$. For generic $[c] \in C$, $\check{[c]}_C = 2([c]) + \sum_{[b] \in C_2} ([b - \frac{c}{2}])$.

Proposition 7.1. *For all* $0 \neq m \in \mathbb{Z}(C)$, $R_{\check{C}}(m) = \mathcal{F}$.

Proof. This follows from Proposition 7.2 below. When $m \in \mathbb{Z}$, we give a proof based on Theorem 3.14. The condition that $[ma-n] \in \mathcal{F}$ for all $[a] \in \mathcal{F}$ means that $[n] \in \mathcal{F}$. Fix $[a] \in \mathcal{F}$. For generic $[c] \in C$, we calculate $\alpha(a) = 2\zeta(a-c) + \sum_{b \in \widehat{C_2}} \zeta(a+\frac{c}{2}-b)$. Since $\sum_{b \in \widehat{C_2}} \zeta(z-b) = 2\zeta(2z)$ for all $[z] \in C$, we get $\alpha(a) = 2\zeta(a-c) + 2\zeta(2a+c) = 2\eta(3a)$. If $[n] \in \mathcal{F}$, then $\alpha_{m,[n]}[a] = 2m\eta(3ma-3n) - 2m^2\eta(3a) + \eta(6mn) = 0$. □

The condition $[n] \in \mathcal{F}$ means that the map $C \xrightarrow{h} C$, $h[z] = [mz-n]$, preserves the collinearity on the smooth cubic C : if $[a]$, $[b]$, $[c]$ are three collinear points on C, then $h[a]$, $h[b]$, $h[c]$ are also collinear. The following construction is apparently due to T. Ueda; the second author has learned it from M. Jonsson.

Proposition 7.2. *Let* $C \times C \xrightarrow{\pi} \check{\mathbb{P}}^2$, $\pi(c_1, c_2) := L(c_1, c_2)$. *Note that* $\pi(c,c) = \nu(c)$ *for all* $c \in C$. *If* $C \xrightarrow{h} C$ *preserves the collinearity on* C, *then* (h,h) *induces through* π *a regular self-map* $\check{h}$ *of* $\check{\mathbb{P}}^2$, $\check{h}\pi = \pi(h,h)$. *Moreover,*

1. *The branching curve of* $\check{h}$ *is* $\check{C}$, *and* $J(\check{h}) = \check{\mathbb{P}}^2$.
2. *The ruled surface* $\{(c, L) \in C \times \check{\mathbb{P}}^2 : \ c \in L\}$ *is* $(h, \check{h})$*-invariant.*
3. *The dual* $\check{C}$ *of* C *is* $\check{h}$*-invariant, with* $\check{h}\nu = \nu h$.
4. *The dual* C *of* $\check{C}$ *is* $\check{h}$*-invariant; for all* $c \in C$, $\check{h}(\check{c}) = (h(c))\check{}$.

Proof. The map π can be identified with the quotient map associated to the action on $C \times C$ of the symmetric group Σ_3 generated by the involution $\alpha(a,b) = (b,a)$ and the 3-cycle $\beta(a,b) = (b, -a-b)$. Since (h,h) commutes with α and β, it induces a regular self-map $\check{h}$ of $\check{\mathbb{P}}^2$. The properties of $\check{h}$ are easy to ˇ. □

Corollary 7.3. *The curve* $\check{C}$ *admits nine tangent processes. They extend uniquely to* $\mathbb{P}^2$. *One of them fixes some (all) of the nine cusps of* $\check{C}$.

Example 7.4. Let $C_k = (\sum_j x_j^3 = 3k \prod_j x_j)$, as in subsection 6.3. The normalization of $\check{C}_k$ is $\nu_k[x_j] = [x_j^2 - kx_{j+1}x_{j+2}]$. When $k=0$, $\check{C}_0 = \left(2\sum_j a_j^6 = (\sum_j a_j^3)^2\right)$. The Desboves map g_0 on C_0 induces through ν_0 the tangent process on $\check{C}_0$ that fixes the cusps. We calculate $\check{g}_0[a_j] = [a_j(-3a_j^3 + 2\sum_k a_k^3)]$.

8. Elliptic quartics with two singular points.

8.1. **Normalization.** Given two plane curves Q_1 and Q_2, we write $Q_1 \sim Q_2$ when there exists a Möbius transformation $M \in \mathrm{Aut}(\mathbb{P}^2)$ with $M(Q_1) = Q_2$. The elliptic quartics with two singular points can be represented as follows.

Proposition 8.1. *Let* $\mathbb{C} \xrightarrow{[\cdot]} \mathbb{C}/\Omega = C$ *be an elliptic curve. Given* $(a, b, \tilde{a}) \in \mathbb{C}^3$, *let* $\tilde{b} := -a-b-\tilde{a}$, $c := -2a-b$, *and* $\tilde{c} := -2\tilde{a}-\tilde{b}$. *Assume that* $[a] \neq [\tilde{a}]$, $[a] \neq [\tilde{b}]$, $[b] \neq [\tilde{a}]$, $[b] \neq [\tilde{b}]$, $[c] \neq [\tilde{a}]$, $[c] \neq [\tilde{b}]$, $[\tilde{c}] \neq [a]$, $[\tilde{c}] \neq [b]$, *and* $[a+b] \neq [\tilde{a}+\tilde{b}]$. *Define* $\mathbb{C} \xrightarrow{\Psi} \mathbb{C}^3$, $\Psi = (X, Y, Z)$, *by:*
$X(t) = \sigma^2(t-\tilde{a})\sigma(t-\tilde{b})\sigma(t-\tilde{c})$,
$Y(t) = \sigma^2(t-a)\sigma(t-b)\sigma(t-c)$,
$Z(t) = \sigma(t-a)\sigma(t-b)\sigma(t-\tilde{a})\sigma(t-\tilde{b})$.

Then Ψ induces through $\mathbb{C} \xrightarrow{[\cdot]} C$ and $\mathbb{C}^3 \setminus 0 \xrightarrow{[\cdot]} \mathbb{P}^2$ a normalization $C \xrightarrow{\nu} Q$, $\nu[t] = [\Psi(t)]$, of an elliptic quartic $Q := C(a, b, \tilde{a})$ with two ordinary singularities, $q := [1, 0, 0]$, $\tilde{q} := [0, 1, 0]$, and $s(\nu^*\mathcal{O}_Q(1)) = 0$, $\nu^*q = ([a]) + ([b])$, $\nu^*\tilde{q} = ([\tilde{a}]) + ([\tilde{b}])$.

All elliptic quartics with two singularities are Möbius images of such $C(a, b, \tilde{a})$.

Proof. It is clear that $X(t)$, $Y(t)$ and $Z(t)$ do not vanish simultaneously. For $\omega \in \Omega$, $\frac{X(t+\omega)}{X(t)} = \frac{Y(t+\omega)}{Y(t)} = \frac{Z(t+\omega)}{Z(t)} = \exp(2(2t + \omega)\eta(\omega))$, so $C \xrightarrow{\nu} \mathbb{P}^2$ is well-defined. In affine coordinates, $x[t] = \frac{X(t)}{Z(t)} = \frac{\sigma(t-\tilde{a})\sigma(t-\tilde{c})}{\sigma(t-a)\sigma(t-b)}$, and $y[t] = \frac{Y(t)}{Z(t)} = \frac{\sigma(t-a)\sigma(t-c)}{\sigma(t-\tilde{a})\sigma(t-\tilde{b})}$. We show that ν is injective on $C \setminus \nu^{-1}(Z = 0)$. Indeed, if $\nu[t_1] = \nu[t_2] \notin (Z = 0)$ for some $[t_1] \neq [t_2]$, then $[t_1] + [t_2]$ would be a fiber of both $x[t]$ and $y[t]$, hence $[a] + [b] = [t_1] + [t_2] = [\tilde{a}] + [\tilde{b}]$. Therefore, $C \xrightarrow{\nu} Q \subset \mathbb{P}^2$ is a normalization map. Clearly, $\nu^*(q) = ([a]) + ([b])$, $\nu^*(\tilde{q}) = ([\tilde{a}]) + ([\tilde{b}])$, and $s(\nu^*\mathcal{O}_Q(1)) = s((Z = 0)_C) = 0$. In local coordinates, we see that q and $\tilde{q}$ are ordinary singularities.

Given any elliptic quartic Q with singularities $q \neq \tilde{q}$, fix a normalization $C \xrightarrow{\nu} Q$. Note that $m_Q(q) = 2 = m_Q(\tilde{q})$, and $L(q, \tilde{q})_C = \nu^*(q) + \nu^*(\tilde{q})$. Choose the group structure $\mathbb{C} \xrightarrow{[\cdot]} C$ so that $\nu^*q = [a] + [b]$, $\nu^*\tilde{q} = [\tilde{a}] + [\tilde{b}]$, with $a + b + \tilde{a} + \tilde{b} = 0$. Choose coordinates in $\mathbb{P}^2$ so that $L(q, \tilde{q}) = (Z = 0)$, $T_{\tilde{q}}[\tilde{a}] = (X = 0)$, and $T_q[a] = (Y = 0)$. Then $(Z = 0)_C = ([a]) + ([b]) + ([\tilde{a}]) + ([\tilde{b}])$, $(X = 0)_C = 2([\tilde{a}]) + ([\tilde{b}]) + ([-2\tilde{a} - \tilde{b}])$, $(Y = 0)_C = 2([a]) + ([b]) + ([-2a - b])$. Rescaling x and y, we get $Q \sim C(a, b, \tilde{a})$. □

Remark 8.2. Let $Q = C(a, b, \tilde{a})$ be an elliptic quartic, with $\tilde{b} := -a - b - \tilde{a}$. Then $Q \sim C(\tilde{a}, \tilde{b}, a) \sim C(b, a, \tilde{a}) \sim C(a, b, \tilde{b})$, $Q \sim C(a + \omega, b + \lambda, \tilde{a} + \tilde{\omega})$ for all $(\omega, \lambda, \tilde{\omega}) \in \Omega^3$, and $Q \sim C(ma - n, mb - n, m\tilde{a} - n)$ for all $(m, [n]) \in \mathbb{U}(C) \times C_4$.

Proof. The first equivalences follow immediately from the proof of Proposition 8.1. For the last one, since $\frac{\sigma(mt)}{\sigma(t)} = m$ for all $t \in \mathbb{C}$, there exist constants $k_i \neq 0$ so that the automorphism $(x, y) \mapsto (k_1 x, k_2 y)$ maps Q onto $C(ma - n, mb - n, m\tilde{a} - n)$. □

Remark 8.3. When q is a node of $Q := C(a, b, \tilde{a})$, we have $\alpha(a, b) = \frac{\sigma(a-\tilde{a})\sigma(a-\tilde{b})}{\sigma(b-\tilde{a})\sigma(b-\tilde{b})}$. When q is a cusp of Q, $\alpha(a) = \zeta(a - \tilde{a}) + \zeta(a - \tilde{b})$. Similarly for $\tilde{q}$.

Proof. In Definition 3.10, choose $l = (X = 0)$ for q, and $l = (Y = 0)$ for $\tilde{q}$. □

When $\mathrm{Aut}(Q)$ is large, the invariants $\alpha_{m,[n]}$ can be calculated more explicitly.

8.2. **Invariant nodal quartics.** A node of a plane curve is *inflectional* iff it is a flex on each of its branches. A *Cassini* curve is an elliptic quartic with two inflectional nodes. In this subsection, we consider a 2-dimensional space of elliptic quartics with two nodes, that contains the Cassini curves.

Remark 8.4. Given an elliptic quartic Q with two nodes, $\mathrm{Aut}(Q) = \mathrm{Aut}_Q(\mathbb{P}^2)$. Moreover, $r_Q(1) > 1$ iff $Q \sim C(a, a - \frac{\lambda}{2}, -a)$, with $[\lambda] = 0 \neq [\frac{\lambda}{2}]$ and $4[a] \neq 0$. Finally, Q is a Cassini curve iff $Q \sim C(\frac{\lambda}{8}, -\frac{3\lambda}{8}, -\frac{\lambda}{8})$, with $[\lambda] = 0 \neq [\frac{\lambda}{2}]$.

Proof. We may assume that $Q = C(a, b, \tilde{a})$. As usual, $\tilde{b} := -a - b - \tilde{a}$. Let $g \in \mathrm{Aut}(Q)$ be induced through ν by $C \xrightarrow{h} C$, $h[t] = [-t - n]$. Since h leaves invariant the set $\{[a], [b], [\tilde{a}], [\tilde{b}]\}$, we get $4[n] = 0$. By Remark 8.2, $g \in \mathrm{Aut}_Q(\mathbb{P}^2)$.

When $m = 1$ and $[n] \neq 0$, replacing if necessary g by g^2 we may assume that $[a - n] = [b]$, $[b - n] = [a]$, $[\tilde{a} - n] = [\tilde{b}]$. We get $2[n] = 0$, $b = a - \frac{\lambda}{2}$ with $[\lambda] = 0$, and $\tilde{a} = -a + \frac{\omega}{2}$ with $[\omega] = 0$. Let $a' := a - \frac{\omega}{4}$. By Remark 8.2, $Q \sim C(a', a' - \frac{\lambda}{2}, -a')$.

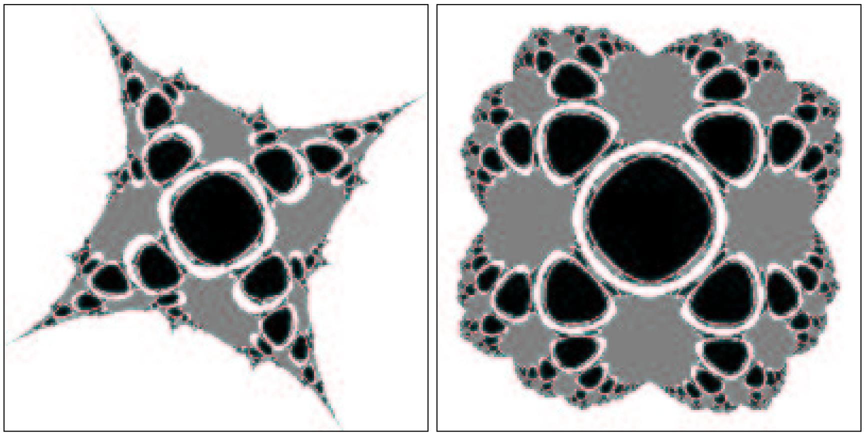

FIGURE 3. Tangent process on Cassini quartic.

When Q is Cassini, $0 = [3a+b] = [3b+a] = [3\tilde{a}+\tilde{b}] = [3\tilde{b}+\tilde{a}]$. We get $8[a]=0$, $[b] = -3[a]$, $2[a+\tilde{a}] = 0$. By Remark 8.2, we may assume that $a = \frac{\lambda}{8}$ with $[\lambda] = 0$, $b = -3a$, and $\tilde{a} = -a + \frac{\omega}{2}$ with $[\omega] = 0$. As before, we may assume that $\omega = 0$. ☐

Proposition 8.5. *Let* $Q = C(a, a - \frac{\lambda}{2}, -a)$, *as in Remark 8.4. If* $0 \neq m \in 2\mathbb{Z}$, *then* $R_Q(m) = \{[n] \in C_{4m} :\ \det(4mn, \lambda) \in 2\mathbb{Z}\}$, *hence* $r_Q(m) = 8m^2$.

Proof. By Remark 8.3, $\alpha(a, a-\frac{\lambda}{2}) = -\exp\left((2a - \frac{\lambda}{2})\eta(\lambda)\right)$. Fix $[n] \in C_{4m}$. By Remark 3.4, $\alpha(ma-n, ma-n-\frac{m}{2}\lambda) = \exp\left((2m^2a - 2mn - \frac{1}{2}m^2\lambda)\eta(\lambda)\right)$. Define $\delta := \det(4mn, \lambda)$. It follows that $\alpha_{m,n}(a, a-\frac{\lambda}{2}) = \exp(\pi \mathrm{i}\delta)$. Similarly, we calculate $\alpha_{m,n}(-a, -a+\frac{\lambda}{2}) = \exp(-\pi\mathrm{i}\delta)$. By Theorem 3.14, $[n] \in R_Q(m)$ iff $\delta \in 2\mathbb{Z}$. ☐

Example 8.6. Given a Cassini quartic Q, there exists a complex number $0 \neq k \neq 1$ so that $Q \sim Q_k := (q_k = 0)$, where $q_k := (1-x^2)(1-y^2) - (1-k)$. Consider the self-map g of Q_k that associates to $p \in Q_k$ the residual intersection $g(p)$ of Q_k with the conic Γ that passes through q, $\tilde{q}$ and p, with tangents $T_q(\Gamma) = (y=1)$ and $T_p(\Gamma) = T_p(Q_k)$. Proposition 8.5 ensures that g is a tangent process on Q_k. We calculate $g[x,y,z] = [2xy(x^2+y^2-2kz^2), x^4-y^4, -2xy(x^2-y^2)]$. To obtain regular extensions f of g to $\mathbb{P}^2$, we can add aq_k to the last component of g, with $0 \neq a \neq \pm 4\mathrm{i}$. In the "limit" situation $k = 0 = a$, we get the degenerate map $(x,y) \mapsto \left(-\frac{x^2+y^2}{x^2-y^2}, -\frac{x^2+y^2}{2xy}\right)$, whose image is the rational quartic $(x^2+y^2 = x^2y^2)$. When $a^2 = 8k$, the restriction of f to the invariant line $(x=0)$ is critically finite. When a and k are real, $\mathbb{PR}^2$ is f-invariant. Figure 3 shows the traces on $\mathbb{R}^2$ (with center at $(0,0)$) of the basins of attraction of $(0,0)$ (black) and $(0, \frac{a}{2})$ (grey), for $a = \sqrt{8k}$, with $k = 0.125$ (left) and $k = 0.001$ (right). Computer experiments suggest that $J(f)$ equals the boundary of the basin of $(0,0)$. However, the unbounded component of $Q_k \cap \mathbb{R}^2$ seems to have a basin of attraction in $\mathbb{R}^2$ (white in Figure 3).

8.3. **Quartics with a cusp and a node.** Let $C^\tau := \mathbb{C}/\Omega_\tau$, with $\Omega_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$. Given a basis (λ, ω) in Ω_τ, let $C^\tau_{\lambda,\omega} := C^\tau(\frac{\lambda+\omega}{8}, \frac{\lambda+\omega}{8}, \frac{\lambda-3\omega}{8})$.

Remark 8.7. Given an elliptic quartic Q with a cusp and a node, $r_Q(-1) > 0$ iff the node of Q is inflectional iff $Q \sim C^\tau_{\lambda,\omega}$ with $(\lambda,\omega) \in \{(1,\tau), (\tau, \tau+1), (\tau+1, 1)\}$.

Proof. By Proposition 8.1 and Remark 8.2, we may assume that $Q = C^\tau(a, a, \tilde{a})$, with $2[a+\tilde{a}] \neq 0 \neq 4[a]$, $[a] \neq [\tilde{a}] \neq -3[a]$. Let $Q \xrightarrow{g} Q$ be induced through ν by $[t] \mapsto [n-t]$. Since g fixes q, $[n] = 2[a]$. Since g must fix the branches of $\tilde{q}$, $2[\tilde{a}] = 2[a]$ and $8[a] = 0$, i.e. $\tilde{q}$ is inflectional. With $\omega := 2(a - \tilde{a})$ and $\lambda := 8a - \omega$, we get $Q = C^\tau_{\lambda,\omega}$. By Remark 8.2, $C^\tau_{\lambda,\omega} \sim C^\tau_{\omega,\lambda}$, and $C^\tau_{\lambda+2\alpha,\omega} \sim C^\tau_{\lambda,\omega} \sim C^\tau_{\lambda,\omega+2\alpha}$ for all $\alpha \in \Omega_\tau$, hence we may assume $(\lambda, \omega) \in \{(1,\tau), (\tau, \tau+1), (\tau+1, 1)\}$. Note that $[t] \mapsto [\frac{\lambda+\omega}{4} - t]$ induces through ν an automorphism of $C^\tau_{\lambda,\omega}$ that extends to $\mathbb{P}^2$. □

Lemma 8.8. *Let $Q = C^\tau_{\lambda,\omega}$, with $a = \frac{\lambda+\omega}{8}$, $\tilde{a} = \frac{\lambda-3\omega}{8}$ and $\tilde{b} = \frac{\omega-3\lambda}{8}$. Then $\alpha(a) = \frac{1}{2}\eta(\lambda+\omega)$, and $\alpha^2(\tilde{a},\tilde{b}) = -\frac{\exp(\frac{\omega}{2}\eta(\omega))}{\exp(\frac{\lambda}{2}\eta(\lambda))} \cdot \frac{\mathcal{P}(\frac{\omega}{2})+2\mathcal{P}(\frac{\lambda}{2})}{\mathcal{P}(\frac{\lambda}{2})+2\mathcal{P}(\frac{\omega}{2})}$.*

Proof. By Remark 8.3, $\alpha(a) = \zeta(\frac{\omega}{2}) + \zeta(\frac{\lambda}{2}) = \frac{1}{2}(\eta(\omega) + \eta(\lambda)) = \frac{1}{2}\eta(\omega+\lambda)$, and $\alpha(\tilde{a},\tilde{b}) = \frac{\sigma^2(\frac{\omega}{2})}{\sigma^2(\frac{\lambda}{2})}$. Since $\mathcal{P}'(t) = \frac{\sigma(2t)}{\sigma^4(t)}$ and $\lim_{t\to 0} \frac{\mathcal{P}'(t+\frac{\lambda}{2})}{\mathcal{P}'(t+\frac{\omega}{2})} = \frac{\mathcal{P}''(\frac{\lambda}{2})}{\mathcal{P}''(\frac{\omega}{2})}$, we obtain $\alpha^2(\tilde{a},\tilde{b}) = \frac{\exp(\frac{\omega}{2}\eta(\omega))}{\exp(\frac{\lambda}{2}\eta(\lambda))} \cdot \frac{\mathcal{P}''(\frac{\lambda}{2})}{\mathcal{P}''(\frac{\omega}{2})}$. The three finite critical values of $\mathcal{P}$ are $e_1 := \mathcal{P}(\frac{\omega}{2})$, $e_2 := \mathcal{P}(\frac{\lambda}{2})$ and $e_3 := -e_1 - e_2 = \mathcal{P}(\frac{\lambda+\omega}{2})$. Recall that $(\mathcal{P}')^2 = -4\prod_i(\mathcal{P} - e_i)$, hence $\mathcal{P}'' = -2\sum_i(\mathcal{P} - e_{i-1})(\mathcal{P} - e_{i+1})$. We get $\mathcal{P}''(\frac{\lambda}{2}) = -2(e_2 - e_1)(e_2 - e_3)$ and $\mathcal{P}''(\frac{\omega}{2}) = -2(e_1 - e_2)(e_1 - e_3)$, hence $\frac{\mathcal{P}''(\frac{\lambda}{2})}{\mathcal{P}''(\frac{\omega}{2})} = -\frac{e_1+2e_2}{e_2+2e_1}$. □

Given $r \in \mathbb{R}$, let $\lfloor r \rfloor$ denote the largest integer less than or equal to r.

Proposition 8.9. *Let $Q = C^\tau_{\lambda,\omega}$, as in Remark 8.7. Given $m \in \mathbb{Z}$, $R_Q(m) \neq \emptyset$ iff $\left(\frac{\mathcal{P}(\frac{\lambda}{2})+2\mathcal{P}(\frac{\omega}{2})}{\mathcal{P}(\frac{\omega}{2})+2\mathcal{P}(\frac{\lambda}{2})}\right)^{\lfloor \frac{m^2}{2} \rfloor} = 1$, in which case $R_Q(m) = \{[(m-1)a]\}$.*

Proof. Fix $[n] \in R_Q(m)$. Since $[t] \mapsto [mt - n]$ must fix $[a]$, we may assume that $n = (m-1)a$, and then $4m[n] = 0$. By Lemma 8.8, $\alpha_{m,n}(a) = 0$.

When m is even, we have $[m\tilde{a} - n] = \left[m\tilde{b} - n\right]$, and Remark 3.4 implies that $\alpha(m\tilde{a} - n, m\tilde{b} - n) = \exp(\frac{1}{4}m(1-2m)(\lambda+\omega)\eta(\lambda-\omega))$. Using Lemma 8.8, we calculate $\alpha_{m,n}(\tilde{a},\tilde{b}) = \left(\frac{\mathcal{P}(\frac{\lambda}{2})+2\mathcal{P}(\frac{\omega}{2})}{\mathcal{P}(\frac{\omega}{2})+2\mathcal{P}(\frac{\lambda}{2})}\right)^{\frac{m^2}{2}}$.

When m is odd, we have $m\tilde{a} - n = \tilde{a} - p\omega$ and $m\tilde{b} - n = \tilde{b} - p\lambda$, with $p := \frac{m-1}{2}$. By Remark 3.4, $\alpha(m\tilde{a} - n, m\tilde{b} - n) = (-1)^p \alpha(\tilde{a},\tilde{b}) \exp(\frac{1}{2}p(4p+3)(\omega\eta(\omega) - \lambda\eta(\lambda)))$. By Lemma 8.8, $\alpha_{m,n}(\tilde{a},\tilde{b}) = \left(\frac{\mathcal{P}(\frac{\lambda}{2})+2\mathcal{P}(\frac{\omega}{2})}{\mathcal{P}(\frac{\omega}{2})+2\mathcal{P}(\frac{\lambda}{2})}\right)^{\frac{m^2-1}{2}}$. □

Corollary 8.10. *Up to Möbius equivalence, $C^{\mathrm{i}}_{1,\mathrm{i}}$ is the only elliptic quartic with a cusp and an inflectional node that admits a tangent process.*

Proof. The condition $\left(\frac{\mathcal{P}(\frac{\lambda}{2})+2\mathcal{P}(\frac{\omega}{2})}{\mathcal{P}(\frac{\omega}{2})+2\mathcal{P}(\frac{\lambda}{2})}\right)^2 = 1$ means $\mathcal{P}(\frac{\lambda+\omega}{2}) = 0$. In this case, C^τ is isomorphic to $(y^2 = x^3 - x)$, i.e. $\tau = \mathrm{i}$. Since $\mathcal{P}(\mathrm{i}t) = -\mathcal{P}(t)$, we have $\mathcal{P}(\frac{1+\mathrm{i}}{2}) = 0$, hence $[\frac{\lambda+\omega}{2}] = [\frac{1+\mathrm{i}}{2}]$. By Remark 8.7, we may assume $(\lambda,\omega) = (1,\mathrm{i})$. □

Example 8.11. Let $Q := (h = 0)$, with $h := (x^2 - y^2)z^2 - xy^3$. Then $Q \sim C^{\mathrm{i}}_{1,\mathrm{i}}$, with cusp $q = [1,0,0]$ and node $n := [0,0,1]$. Let $r := [0,1,0] \in T_q(Q) \cap Q$. The tangent process on Q maps $p \in Q$ to the residual intersection $g(p)$ of Q with the conic that passes through q, n, r, p, and is tangent to Q at p. We calculate $g[x,y,z] = [4(xy+2z^2)^2 - (x^2+y^2)^2, 16x^2z^2 - 4xy(x^2-y^2), 8xz(x^2+y^2)]$.

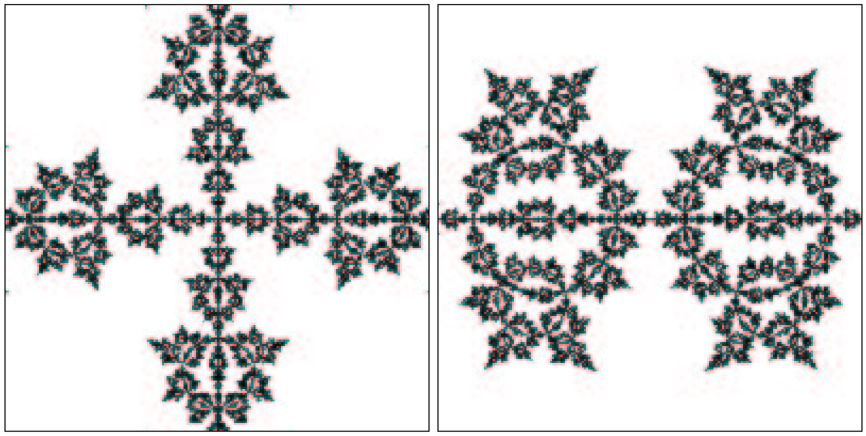

FIGURE 4. Tangent process on "mixed" quartic.

The group $\mathrm{Aut}(Q)$ is generated by $[x,y,z] \mapsto [x,-y,\mathrm{i}z]$. The regular extensions of g that commute with $\mathrm{Aut}(Q)$ are obtained by adding ah to the second component of g, with $0 \neq a$. When $a = \frac{16b}{(b-1)(b+3)}$, with $|b^2-1| < 1$ and $|b^2+1| < 2$, the fixed points $\left[\pm\sqrt{\frac{b-1}{b+3}}, 1, 0\right]$ are attracting for the self-map f_b obtained in this way; by Remark 5.5, $J(f_b)$ is the common boundary of their basins. Figure 4 shows the traces of $J(f_{-0.9})$ on the lines $(x=0)$ (left) and $(z=0)$ (right), near $[0,1,0]$.

8.4. **Invariant cuspidal quartic.** We discuss now the invariance of the elliptic quartics with two cusps.

Remark 8.12. Given an elliptic quartic Q with two cusps, $Q \sim C(a) := C(a,a,-a)$, with $4[a] \neq 0$. Moreover, $\mathrm{Aut}(Q) = \mathrm{Aut}_Q(\mathbb{P}^2)$, and $r_Q(-1) = 1$. The tangents to Q at the cusps meet at a flex of Q iff $Q \sim C^\rho(\frac{1+\rho}{6})$, with $\rho := \exp(\frac{\pi \mathrm{i}}{3})$.

Proof. The first statement follows from Proposition 8.1 and Remark 8.2. Let $g \in \mathrm{Aut}(Q)$, with $Q = C(a)$, be induced through ν by $C \xrightarrow{h} C$, $h[t] = [mt-n]$. Since h leaves invariant the set $\{[a], -[a]\}$, we get $2[n] = 0$. By Remark 8.2, $g \in \mathrm{Aut}_Q(\mathbb{P}^2)$. When $m = -1$, we must have $h[a] = -[a]$, i.e. $[n] = 0$.

The tangents to $C(a)$ at the two cusps meet at $r := [0,0,1]$. Clearly, $r \in C(a)$ iff $6[a] = 0$. In this case, $r = \nu[3a]$, and r is a flex of $C(a)$ iff $\left(\frac{Y}{X}\right)'(3a) = 0$, i.e. $3\zeta(2a) = \eta(6a)$. By the addition theorems of ζ and $\mathcal{P}$, this happens iff $\mathcal{P}''(2a) = 0$ iff $\mathcal{P}(2a) = 0$. Assuming this, the second-order differential equation of $\mathcal{P}$ implies that $\sum_i e_{i-1}e_{i+1} = 0$, where e_i are the three finite critical values of $\mathcal{P}$. Therefore, C is isomorphic to C^ρ, in which case $\mathcal{P}(\rho t) = -\rho\mathcal{P}(t)$ for all t, hence $\mathcal{P}(\frac{1+\rho}{3}) = 0$. It follows that $2[a] = \pm[\frac{1+\rho}{3}]$, and we may assume $a = \frac{1+\rho}{6}$, by Remark 8.2. □

Proposition 8.13. *Let $Q = C(a)$, as in Remark 8.12. Given $m \in \mathbb{Z}$, $R_Q(m) \neq \emptyset$ iff there exists $\epsilon \in \{-1,0,1\}$ so that $2(m-\epsilon)[a] = 0$ and $(m-\epsilon)\zeta(2a) = \eta(2(m-\epsilon)a)$. When $\epsilon = 0$, $R_Q(m) = \{(m \pm 1)[a]\}$; when $\epsilon = \pm 1$, $R_Q(m) = \{(m-\epsilon)[a]\}$.*

Proof. Given $n \in R_Q(m)$, let $Q \xrightarrow{g} Q$ be induced through ν by $[t] \mapsto [mt-n]$. Recall that $q = \nu[a]$ and $\tilde{q} = \nu[-a]$ are the cusps of Q. There are three possibilities.

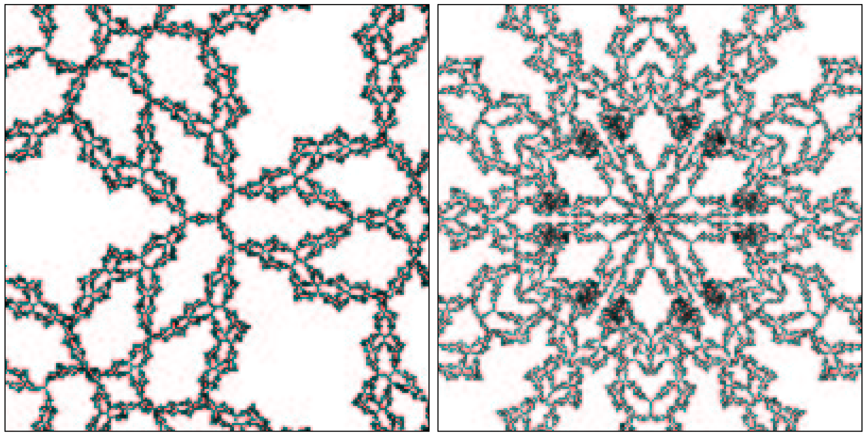

FIGURE 5. Tangent process on cuspidal quartic.

1. $g(q) = q$ and $g(\tilde{q}) = \tilde{q}$, i.e. $2(m-1)[a] = 0$ and $[n] = (m-1)[a]$.
2. $g(q) = g(\tilde{q})$, i.e. $2m[a] = 0$ and $[n] = (m \pm 1)[a]$.
3. $g(q) = \tilde{q}$ and $g(\tilde{q}) = q$, i.e. $2(m+1)[a] = 0$ and $[n] = (m+1)[a]$.

Therefore, $2(m-\epsilon)a = \omega \in \Omega$, for a (unique) $\epsilon \in \{-1, 0, 1\}$. When $\epsilon = \pm 1$, we may assume $n = (m-\epsilon)a$. When $\epsilon = 0$, we may assume $n = (m-1)a$ (conjugate if necessary g with the involution induced by $[t] \mapsto [-t]$).

By Remark 8.3, $\alpha(a) = 2\zeta(2a) = -\alpha(-a)$.

Assume $\epsilon = \pm 1$. Then $\alpha(ma-n) = \epsilon\alpha(a)$, $\alpha_{m,n}(a) = 2m\,(\eta(\omega) - (m-\epsilon)\zeta(2a))$, $\alpha(-ma-n) = \alpha(-\epsilon a - \omega) = -\epsilon\alpha(a) - 4\eta(\omega)$, and $\alpha_{m,n}(-a) = -\alpha_{m,n}(a)$.

Assume $\epsilon = 0$. Then $\alpha(ma-n) = \alpha(a)$, $\alpha_{m,n}(a) = 2(m-1)\,(\eta(\omega) - m\eta(2a))$, $\alpha(-ma-n) = \alpha(a-\omega) = \alpha(a) - 4\eta(\omega)$, $\alpha_{m,n}(-a) = -2(m+1)(\eta(\omega) - m\eta(2a))$. □

Corollary 8.14. *Up to Möbius equivalence, $C^\rho(\frac{1+\rho}{6})$ is the only elliptic quartic with two cusps that admits a tangent process.*

Proof. By Proposition 8.13, $6[a] = 0$ and $3\zeta(2a) = \eta(6a)$. (The proof of) Remark 8.12 finishes the proof. □

Example 8.15. Let $Q := (h = 0)$, with $h := (y^2 - z^2)^2 - 8x^3 z$. Then $Q \sim C^\rho(\frac{1+\rho}{6})$; the tangents to Q at the cusps $q := [0,1,1]$, $\tilde{q} := [0,-1,1]$ meet at the flex $r := [1,0,0]$. The tangent process on Q maps $p \in Q$ to the residual intersection $g(p)$ of Q with the conic that passes through q, $\tilde{q}$, r, p, and is tangent to Q at p. We calculate $g[x,y,z] = [x(x^3 - 2y^2 z + 10z^3), y(7z^3 - 5y^2 z - 2x^3), 2y^4]$.

The group $\mathrm{Aut}(Q)$ is generated by $[x,y,z] \mapsto [\rho x, y, -z]$. The extensions f_a of g that commute with $\mathrm{Aut}(Q)$ are obtained by adding ah to the last component of g. When $3.484 < a < 10$, f_a has three attracting points on the invariant line $(y = 0)$, and, by Remark 5.5, $J(f_a)$ is the common boundary of their basins. Figure 8.4 shows the traces of $J(f_4)$ on the lines $(z = 0.01y)$ (left) and $(z = 0)$ (right), near r.

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