MTH - 244 EXAM I-A Spring 2008

Name: Araceli Student ID:

Section: Instructor:

Show all your work.! NO WORK, NO CREDIT!

I. (15 pts.) Solve the initial value problem

$$9y'' + 6y' + 82y = 0, \qquad y(0) = -1, \quad y'(0) = 2$$

$9r^2+6r+82=0$

$r = \dfrac{-6 \pm \sqrt{36-4(9)(82)}}{2(9)} = \dfrac{-6 \pm \sqrt{-2916}}{18} = \dfrac{-6 \pm 54i}{18}$

$r = -\dfrac{1}{3} \pm 3i$

$\boxed{r_1 = -\frac{1}{3} + 3i \quad r_2 = -\frac{1}{3} - 3i}$

$\therefore\ y = c_1 e^{(-1/3+3i)t} + c_2 e^{(-1/3-3i)t}$

$y(t) = e^{-1/3 t}\left((c_1+c_2)\cos 3t + (c_1-c_2)\sin 3t\right)$

$-1 = y(0) = c_1 + c_2 \Rightarrow c_1 = -1-c_2$

$y'(t) = -\frac{1}{3} e^{-1/3 t}\left((c_1+c_2)\cos 3t + (c_1-c_2)\sin 3t\right)$

$+ e^{-1/3 t}\left(-3(c_1+c_2)\sin 3t + 3(c_1-c_2)\cos 3t\right)$

$2 = y'(0) = -\frac{1}{3}(c_1+c_2) + 3i(c_1-c_2)$

$\longrightarrow \quad \longrightarrow$

Since $c_1 = -1 - c_2$

$$2 = -\frac{1}{3}\left(-1 - \cancel{c_2} + \cancel{c_2}\right) + 3i\left(-1 - c_2 - c_2\right)$$

$$2 = \frac{1}{3} - 3i - 6ic_2$$

$$2 - \frac{1}{3} + 3i = -6ic_2$$

$$\frac{5}{3} + 3i = -6ic_2$$

$$\frac{5i}{-18ii} + \frac{3\cancel{i}}{-6\cancel{i}} = c_2$$

$$\frac{5i}{18} - \frac{1}{2} = c_2$$

Since

$$c_1 = -1 - c_2$$

$$= -1 - \left(-\frac{1}{2} + \frac{5i}{18}\right)$$

$$= -1 + \frac{1}{2} - \frac{5i}{18}$$

$$c_1 = -\frac{1}{2} - \frac{5i}{18}$$

$$c_1 + c_2 = -\frac{1}{2} \cancel{-\frac{5i}{18}} - \frac{1}{2} \cancel{+\frac{5i}{18}} = -1$$

$$c_1 - c_2 = -\cancel{\frac{1}{2}} - \frac{5i}{18} + \cancel{\frac{1}{2}} - \frac{5i}{18} = -\frac{10i}{18} = -\frac{5}{9}i$$

$$\therefore y(t) = + e^{-\frac{1}{3}t}\left(-\cos 3t + \frac{5}{9} \sin 3t\right)$$

An Easier Method to Find c_1 & c_2

write. $r = -\frac{1}{3} \pm 3i$

$$y(t) = e^{-\frac{1}{3}t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$y'(t) = -\frac{1}{3} e^{-\frac{1}{3}t}(c_1 \cos 3t + c_2 \sin 3t) +$$
$$e^{-\frac{1}{3}t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$-1 = y(0) = c_1 \Rightarrow c_1 = -1$$

$$2 = y'(0) = -\frac{1}{3} c_1 + 3c_2 \Rightarrow 2 = \frac{1}{3} + 3c_2$$

$$2 - \frac{1}{3} = 3c_2$$

$$\frac{5}{3} = 3c_2$$

$$\frac{5}{9} = c_2$$

$$y(t) = e^{-\frac{1}{3}t}\left(-\cos 3t + \frac{5}{9} \sin 3t\right)$$

II. (15 pts.) Solve the initial value problem

$$y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1$$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$\therefore\ y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

$2 = y(-1) = c_1 e^{2} - c_2 e^{2} \Rightarrow c_1 = \dfrac{2 + c_2 e^2}{e^2}$

$y'(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$

$1 = y'(-1) = -2c_1 e^2 + 2c_2 e^2 + c_2 e^2$

$1 = -2c_1 e^2 + 3c_2 e^2$

$1 = -2\left(\dfrac{2 + c_2 e^2}{\cancel{e^2}}\right)\cancel{e^2} + 3c_2 e^2$

$1 = -4 - 2c_2 e^2 + 3c_2 e^2$

$5 = c_2 e^2 \Rightarrow c_2 = 5e^{-2}$

$\therefore\ c_1 = \dfrac{2+5}{e^2} = 7e^{-2}$

$\therefore\ y(t) = 7e^{-2} e^{-2t} + 5t e^{-2} e^{-2t}$

$y(t) = 7e^{-(2+2t)} + 5t\, e^{-(2+2t)}$

III. (15 pts.) Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

$$y'' + (\cos t)y' + 3(\ln|t|)y = 0, \qquad y(2) = 3, \quad y'(2) = 1$$

Cost is continuous everywhere

$\ln|t|$ is defined for $t > 0$

$\therefore$ The longest interval is $(0, \infty)$

IV. (25 pts.) Find the solution of the given initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \qquad y(\pi) = 0, \quad t > 0$$

$$\mu y' + \frac{2}{t} y\mu = \frac{\cos t}{t^2}\, \mu$$

$$\therefore \frac{d\mu}{dt} = \frac{2\mu}{t}$$

$$\therefore \int \frac{d\mu}{\mu} = \int \frac{2}{t}\, dt$$

$$\ln \mu = 2\ln t + c = \ln t^2 + \ln c = \ln ct^2$$

$$\mu = e^{\ln \mu} = e^{\ln ct^2} = ct^2$$

taking $c=1$

$$\boxed{\mu = t^2}$$

$$\frac{d(t^2 y)}{dt} = \frac{\cos t}{\cancel{t^2}} \cdot \cancel{t^2}$$

$$\int d(t^2 y) = \int \cos t\, dt$$

$$t^2 y = \sin t + C$$

$$y = \frac{\sin t}{t^2} + \frac{C}{t^2}$$

Since $y(\pi) = 0$

$$0 = y(\pi) = \frac{\cancel{\sin \pi}^{\,0}}{\pi^2} + \frac{C}{\pi^2} \Rightarrow C = 0$$

$$\therefore \boxed{y(t) = \frac{\sin t}{t^2}} = \frac{\sin t}{t^2}$$

V. (30 pts.) Find an integrating factor and solve the given equation.

$$dx + (\frac{x}{y} - \sin y)\, dy = 0$$

In case you need them

$$\mu_x = \frac{(M_y - N_x)}{N}\mu \qquad \text{and} \qquad \mu_y = \frac{(N_x - M_y)}{M}\mu$$

$M = 1 \qquad N = \frac{x}{y} - \sin y$

$M_y = 0 \neq N_x = \frac{1}{y}$ Not exact!

$\mu_y = \left(\frac{\frac{1}{y} - 0}{1}\right)\mu$

$\frac{d\mu}{dy} = \frac{\mu}{y}$

$\int \frac{d\mu}{\mu} = \int \frac{dy}{y}$

$\ln \mu = \ln y \Rightarrow e^{\ln \mu} = e^{\ln y}$

$\therefore \mu = y$

$\underbrace{y}_{M}\, dx + (\underbrace{x - y\sin y}_{N})\, dy = 0$

$M_y = 1$, $N_x = 1$ > Exact!

$\therefore \exists\ G(x,y)$ s.t.

$\rightarrow G_x = y$

$G_y = \frac{x}{y} - \sin y$

$\therefore G = \int \frac{\partial G}{\partial x} = yx + h(y)$

$G_y = \cancel{x} + h'(y) = \cancel{x} - y\sin y$

$\Rightarrow h'(y) = -y\sin y$

$\therefore h(y) = \int -y\sin y\, dy$

$u = y \quad dv = -\sin y$

$du = dy \quad v = +\cos y$

$h(y) = y\cos y - \int \cos y\, dy$

$= y\cos y - \sin y$

$\therefore$

$G = yx + y\cos y - \sin y = C$

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I. (15 pts.) Solve the initial value problem

$$9y'' - 12y' + 4y = 0, \qquad y(0) = 2, \quad y'(0) = -1$$

$$9r^2 - 12r + 4 = 0$$

$$r = \frac{12 \pm \cancel{\sqrt{144 - 4(9)(4)}} \to 0}{2(9)}$$

$$r = \frac{12}{18} = \frac{2}{3}$$

$$\therefore y(t) = c_1 e^{2/3 t} + c_2 t e^{2/3 t}$$

$$2 = y(0) = c_1 + \cancel{c_2} \Rightarrow c_1 = 2$$

$$y'(t) = \frac{2}{3} c_1 e^{2/3 t} + c_2 e^{2/3 t} + c_2 \frac{2}{3} t e^{2/3 t}$$

$$-1 = y'(0) = \frac{2}{3} c_1 + c_2$$

since $c_1 = 2$

$$c_2 = -1 - \frac{4}{3} = -\frac{7}{3}$$

$$\therefore y(t) = 2e^{2/3 t} - \frac{7}{3} t e^{2/3 t}$$

II. (15 pts.) Solve the initial value problem

$$4y'' + 4y' + 5y = 0, \qquad y(0) = 3, \qquad y'(0) = 1$$

$$4r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(4)(5)}}{2(4)}$$

$$r = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm 8i}{8} = -\frac{1}{2} \pm i$$

$$r_1 = -\frac{1}{2} + i$$

$$r_2 = -\frac{1}{2} - i$$

$$y(t) = c_1 e^{(-1/2+i)t} + c_2 e^{(-1/2-i)t}$$

$$= e^{-1/2 t}\left((c_1+c_2)\cos t + (c_1-c_2)\sin t\right)$$

$$3 = y(0) = c_1 + c_2 \Rightarrow c_1 = 3 - c_2$$

$$y'(t) = -\frac{1}{2} e^{-1/2 t}\left((c_1+c_2)\cos t + (c_1-c_2)\sin t\right)$$

$$+ e^{-1/2 t}\left(-(c_1+c_2)\sin t + (c_1-c_2)\cos t\right)$$

$$1 = y'(0) = -\frac{1}{2}(c_1+c_2)$$

Maybe it is better to do the following

$$y(t) = e^{-1/2 t}(c_1 \cos t + c_2 \sin t).$$

$$3 = y(0) = c_1$$

$$y'(t) = -\frac{1}{2} e^{-1/2 t}(c_1 \cos t + c_2 \sin t) + e^{-1/2 t}(-c_1 \sin t + c_2 \cos t)$$

$$1 = y'(0) = -\frac{1}{2}c_1 + c_2$$

since $c_1 = 3$

$$c_2 = 1 + \frac{3}{2} = \frac{5}{2}$$

$$\therefore y(t) = e^{-\frac{1}{2}t}\left(3\cos t + \frac{5}{2}\sin t\right)$$

III. (15 pts.) Are the following functions linearly independent (ℓ.i.) or linearly dependent (ℓ.d.). Justify your answer.

$$\cos^2\theta, \qquad 1+\cos 2\theta$$

$f = \cos^2\theta \qquad g = 1+\cos 2\theta$

$f' = -2\cos\theta\sin\theta \qquad g' = -2\sin 2\theta$

$$W = \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ -2\cos\theta\sin\theta & -2\sin 2\theta \end{vmatrix} = \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ -\sin 2\theta & -2\sin 2\theta \end{vmatrix}$$

$$= -2\cos^2\theta\sin 2\theta + \sin 2\theta(1+\cos 2\theta)$$

$$= \sin 2\theta(-2\cos^2\theta + 1 + \cos 2\theta)$$

$$= \sin 2\theta(-2\cancel{\cos^2}\theta + 1 + \cos^2\theta - \sin^2\theta)$$

$$= \sin 2\theta(1 - \cancel{(\cos^2\theta + \sin^2\theta)}) = 0$$

$\underline{l.d}$

IV. (25 pts.) Find the solution of the given initial value problem

$$y' + 2y = te^{-2t}, \qquad y(1) = 0$$

$$\mu y' + 2y\mu = te^{-2t}\mu$$

$$\frac{d\mu}{dt} = 2\mu$$

$$\int \frac{d\mu}{\mu} = \int 2dt$$

$$\ln \mu = 2t + c$$

$$\mu = ce^{2t}$$

take $c=1$

$$\mu = e^{2t}$$

$$\frac{d(e^{2t}\cdot y)}{dt} = t\cancel{e^{-2t}}\cancel{e^{2t}} = t$$

$$\int d(e^{2t}y) = \int t\,dt$$

$$e^{2t}y = \frac{t^2}{2} + c$$

$$y = \left(\frac{t^2}{2} + c\right)e^{-2t}$$

$$0 = y(1) = \left(\frac{1}{2} + c\right)e^{-2}$$

$$\Rightarrow c = -\frac{1}{2}$$

$$\boxed{y(t) = \frac{1}{2}(t^2 - 1)e^{-2t}}$$

V. (30 pts.) Find an integrating factor and solve the given equation

$$\underbrace{y}_{M}\, dx + \underbrace{(2xy - e^{-2y})}_{N}\, dy = 0$$

In case you need them

$$\mu_x = \frac{(M_y - N_x)}{N}\mu \qquad \text{and} \qquad \mu_y = \frac{(N_x - M_y)}{M}\mu$$

$M_y = 1$

$N_x = 2y$ $\neq$ Not exact!

$\mu_x = \frac{(1-2y)}{N}$ not good!

$\mu_y = \left(\frac{2y-1}{y}\right)\mu$

$\frac{d\mu}{dy} = \left(2 - \frac{1}{y}\right)\mu$

$\int \frac{d\mu}{\mu} = \int 2\, dy - \int \frac{1}{y}\, dy$

$\ln \mu = 2y - \ln y$

$e^{\ln \mu} = e^{2y - \ln y} = e^{-\ln y} e^{2y} = y^{-1} e^{2y}$

$\mu = y^{-1} e^{2y}$

$\therefore$ Multiplying the original by μ we get

$e^{2y}\, dx + (2x e^{2y} - y^{-1})\, dy = 0$

$M_y = 2e^{2y}$ Exact! $\exists\, G(x,y)$ s.t.

$N_x = 2e^{2y}$

$G_x = e^{2y}$

$G_y = 2x e^{2y} - y^{-1}$

Integrating w.r.t. x

$G = x e^{2y} + h(y)$

Now

$G_y = \cancel{2x e^{2y}} + h'(y)$

$= \cancel{2x e^{2y}} - y^{-1}$

$\Rightarrow h'(y) = -y^{-1}$

$\therefore h(y) = -\ln y$

$h(y) = \ln y^{-1}$

$G(x,y) = x e^{2y} + \ln y^{-1} = c$

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Show all your work.! NO WORK, NO CREDIT!

I. (15 pts.) Determine a suitable form for the particular solution of the following ODE. **DO NOT SOLVE THE PROBLEM, JUST GIVE THE PARTICULAR SOLUTION.**

$$y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t+4)\sin t$$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2)$

$r_1 = 3$

$r_2 = 2$

$y(t) = c_1 e^{2t} + c_2 e^{3t}$ Solution to homogeneous eq.

Take $g_1(t) = e^{2t}(3t+4)\sin t$

$Y_1(t) = (B_1 + B_2 t)e^{2t}\sin t + (C_1 + C_2 t)e^{2t}\cos t$

Take $g_2(t) = e^t \cos 2t$

$Y_2(t) = e^t(A_1 \cos 2t + A_2 \sin 2t)$

$Y_p(t) = Y_1(t) + Y_2(t)$

II. (15 pts.) Using the method of **VARIATION OF PARAMETERS** find the solution of the given nonhomogeneous equation.

$ty'' - (1+t)y' + y = t^2e^{2t}, \qquad t > 0 \qquad y_1(t) = 1 + t \qquad y_2(t) = e^t$

$g(t) = te^{2t}$

$y_1 = 1+t \quad y_2 = e^t$

$y_1' = 1 \quad y_2' = e^t$

$W = te^t$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = e^t(1+t) - e^t = te^t$$

$$u_1 = -\int \frac{e^t(\cancel{t}e^{2t})}{\cancel{t}e^t}\,dt = -\int e^{2t}\,dt = \cdots - \frac{e^{2t}}{2}$$

$$u_2 = \int \frac{(1+t)(\cancel{t}\cancel{e^{2t}})}{\cancel{t}e^t}\,dt = \int (1+t)\,e^t = (1+t)e^t - \int e^t\,dt = (1+t)e^t - e^t = te^t$$

$u = 1+t \quad dv = e^t$

$du = dt \quad v = e^t$

$$y(t) = -\frac{(1+t)e^{2t}}{2} + te^{2t} = -\frac{e^{2t}}{2} + \frac{te^{2t}}{2}$$

IV. (25 pts.)

(1) Determine the general solution of the given ODE that is valid in any interval not including the singular point.

$$2x^2y'' - 4xy' + 6y = 0$$

$$F(r) = r^2 - 3r + 3$$

$$r = \frac{3 \pm i\sqrt{3}}{2}$$

$$\therefore\ y = |x|^{3/2}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right)$$

(2) Is the singular point regular or irregular? Justify your answer.

$$\left.\begin{array}{l} \lim\limits_{x\to 0} x \cdot \dfrac{-4x}{2x^2} = -2 \\ \lim\limits_{x\to 0} x^2 \cdot \dfrac{6}{2x^2} = 3 \end{array}\right\}\ \text{regular!}$$

V. (30 pts.) Use **Laplace Transform** to solve the given initial value problem.

$$y'' + 2y' + y = 4e^{-t}, \qquad y(0) = 2, \quad y'(0) = -1$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = \frac{4}{s+1}$$

$$(s^2+2s+1)\mathcal{L}\{y\} - 2s \cancel{+1} - 4 = \frac{4}{s+1}$$

$$(s+1)^2\mathcal{L}\{y\} = \frac{4}{s+1} + 3 + 2s = \frac{4 + \cancel{3s} + 3 + 2s^2 + \cancel{2s}}{s+1}$$

$$\mathcal{L}\{y\} = \frac{2s^2+5s+7}{(s+1)(s+1)^2} = \frac{2s^2+5s+7}{(s+1)^3}$$

By partial Fractions

$$\frac{2s^2+5s+7}{(s+1)^3} = \frac{A}{(s+1)^3} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)}$$

$$= \frac{A + B(s+1) + C(s+1)^2}{(s+1)^3} = \frac{A + \cancel{Bs} + B + Cs^2 + \cancel{2Cs} + C}{(s+1)^3}$$

$$= \frac{Cs^2 + s(B+2C) + A + C + B}{(s+1)^3}$$

$\Rightarrow C = 2$

$B + 2C = 5$

$B = 5 - 2C = 5 - 4 = 1$

$A + C + B = 7 \Rightarrow A = 7 - C - B = 7 - 2 - 1 = 4$

$$\therefore \frac{2s^2-5s+7}{(s+1)^3} = \frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}$$

Taking inverse Laplace transf.

$$y(t) = 2t^2e^{-t} + te^{-t} + 2e^{-t}$$

since $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\} = 2t^2e^{-t}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = 2e^{-t}$$

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Show all your work.! NO WORK, NO CREDIT!

I. (15 pts.) Determine a suitable form for the particular solution of the following ODE. **DO NOT SOLVE THE PROBLEM, JUST GIVE THE PARTICULAR SOLUTION.**

$$y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1)$

$r_2 = -2$

$r_1 = -1$

$y(t) = c_1 e^{-t} + c_2 e^{-2t}$ Slution to homogeneous eq.

Let $g_1 = e^t(t^2+1)\sin 2t$.

$Y_1(t) = e^t(A_0 + A_1 t + A_2 t^2)\cos 2t + e^t(B_0 + B_1 t + B_2 t^2)\sin 2t$

Let $g_2 = 3e^{-t}\cos t$

$Y_2(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$

Let $g_3 = 4e^t$

$Y_3(t) = De^t$

$\therefore Y_p(t) = Y_1(t) + Y_2(t) + Y_3(t)$

II. (15 pts.) Using the method of **VARIATION OF PARAMETERS** find the solution of the given nonhomogeneous equation.

$(1-t)y''+ty'-y = 2(t-1)^2e^{-t}, \qquad 0<t<1, \qquad y_1(t)=e^t, \quad y_2(t)=t$

$g(t) = 2(1-t)e^{-t}$

$y_1(t) = e^t \quad y_2(t) = t$

$y_1'(t) = e^t \quad y_2' = 1$

$W = (1-t)e^t$

$W = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = e^t - te^t = (1-t)e^t$

$u_1 = -\int \frac{2t\cancel{(1-t)}\,e^{-t}}{\cancel{(1-t)}e^t}\,dt = 2\int te^{-2t}\,dt$

$u = t \quad dv = e^{-2t}$

$du = dt \quad v = -\frac{1}{2}e^{-2t}$

$= -\frac{t}{2}e^{-2t} + \frac{1}{2}\int e^{-2t}\,dt$

$= -\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t}$

$\Rightarrow u_1 = 2\left(-\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t}\right) = te^{-2t} + \frac{e^{-2t}}{2}$

$u_2 = \int \frac{2\cancel{(1-t)}}{\cancel{(1-t)}e^t}\,dt = 2\int e^{-t}\,dt = -2e^{-t}$

$y(t) = te^{-t} + \frac{e^{-t}}{2} - 2te^{-t} = -te^{-t} + \frac{e^{-t}}{2}$

III. (15 pts.) Solve the given ODE by means of a **power series about the given point** x_0. Find the recurrence relation, also find the first 3 terms in each of the two linearly independent solutions.

$$(4 - x^2)y'' + 2y = 0, \qquad x_0 = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(4-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2\sum_{n=0}^{\infty} a_n x^n$$

$$4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$4 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\cancel{12a_2}\ 8a_2 + 24 a_3 x + 4\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$+ 2a_0 + 2a_1 x + 2\sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow \cancel{12a_2 + 2a_0 = 0 \Rightarrow a_2 =}$$

$$8a_2 + 2a_0 = 0 \Rightarrow a_2 = -\frac{2a_0}{8} = -\frac{a_0}{4}$$

$$24a_3 + 2a_1 = 0 \Rightarrow a_3 = -\frac{2a_1}{24} = -\frac{a_1}{12}$$

Now

$$4(n+2)(n+1)a_{n+2} = (n(n-1) - 2)a_n$$

$$\Rightarrow a_{n+2} = \frac{\cancel{(n+1)}(n-2)}{4(n+2)\cancel{(n+1)}} a_n$$

$$a_{n+2} = \frac{(n-2)}{4(n+2)} a_n$$

$$a_4 = 0$$

$$a_5 = \frac{a_3}{4 \cdot 5} = -\frac{a_1}{20 \cdot 12} = -\frac{a_1}{240}$$

$$\therefore \quad y_1(x) = a_0\left(1 - \frac{x^2}{4}\right)$$

$$y_2(x) = a_1\left(x - \frac{x^3}{12} - \frac{x^5}{240} - \cdots\right)$$

IV. (25 pts.)

(1) Determine the general solution of the given ODE that is valid in any interval not including the singular point.

$$x^2y'' - 3xy' + 4y = 0$$

$$F(r) = r^2 - 4r + 4$$

$$F(r) = (r-2)^2$$

$$\therefore\ y(x) = (c_1 + c_2 \ln|x|)\,x^2$$

(2) Is the singular point regular or irregular? Justify your answer.

$$\left.\begin{aligned} &\lim_{x\to 0} x \cdot \frac{3x}{x^2} = 3 \\ &\lim_{x\to 0} x^2 \cdot \frac{4}{x^2} = 4 \end{aligned}\right\}\ \text{Regular!}$$

V. (30 pts.) Use **Laplace Transform** to solve the given initial value problem.

$$y'' - 2y' + 2y = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - \cancel{sy(0)} - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s^2\mathcal{L}\{y\} - 1 - 2s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$(s^2 - 2s + 2)\,\mathcal{L}\{y\} = \frac{1}{s+1} + 1 = \frac{1+s+1}{s+1} = \frac{s+2}{s+1}$$

$$\mathcal{L}\{y\} = \frac{s+2}{(s+1)(s^2-2s+2)}$$

Partial Fractions

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

$$= \frac{A\cancel{s^2} - 2\cancel{As} + 2A + \cancel{Bs^2} + \cancel{Cs} + Bs + C}{(s+1)(s^2-2s+2)}$$

$$= \frac{s^2(A+B) + s(-2A+C+B) + (2A+C)}{(s+1)(s^2-2s+2)}$$

$$A+B=0 \Rightarrow A=-B \Rightarrow B=-A \qquad B=-\frac{1}{5}$$

$$-2A+C+B=1 \Rightarrow -2\cancel{A}+2-2\cancel{A}-\cancel{A}=1 \Rightarrow -5A=-1 \qquad A=\frac{1}{5}$$

$$2A+C=2 \Rightarrow C=2-2A$$

$$C = 2 - \frac{2}{5} = \frac{10-2}{5} = \frac{8}{5}$$

$$\therefore \frac{s+2}{(s+1)(s^2-2s+2)} = \frac{1/5}{s+1} + \frac{-1/5\,s + 8/5}{s^2-2s+2}$$

$$= \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{5} \frac{(s-8)}{(s^2-2s+2)}$$

Now $s^2-2s+2=(s-1)^2+1$

therefore $\dfrac{s-8}{(s-1)^2+1} = \dfrac{s-1-7}{(s-1)^2+1} = \dfrac{(s-1)}{(s-1)^2+1} - \dfrac{7}{(s-1)^2+1}$

We get

$$= \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{5} \frac{(s-1)}{(s-1)^2+1} + \frac{7}{5} \cdot \frac{1}{(s-1)^2+1}$$

Taking inverse Laplace transf

$$y(t) = \frac{1}{5} e^{-t} - \frac{1}{5} e^{t} \cos t + \frac{7}{5} \cdot e^{t} \sin t$$